Correlated-belief equilibrium*

Elias Tsakas†

Maastricht University

June 2, 2015

Abstract

We introduce a new solution concept, called correlated-belief equilibrium. The difference to Nash equilibrium is that, while each player has correct marginal conjectures about each opponent, it is not necessarily the case that these marginal conjectures are independent. Then, we provide an epistemic foundation and we relate correlated-belief equilibrium with standard solution concepts, such as rationalizability, correlated equilibrium and conjectural equilibrium.

1. Introduction

Beliefs have recently become a very useful tool for game-theoretic analysis. One of the main advantages from incorporating beliefs into our game-theoretic models is that they allow us to explicitly distinguish between what players think that their opponents will do and what their opponents actually do. Thus, they help us better understand the implicit assumptions that are often present in the definition of a solution concept. Take for instance Nash Equilibrium (NE), which is the most well-known and widely-used game-theoretic solution concept. According to the standard definition, a strategy profile is a NE if each player's strategy is optimal given the opponents' strategies (Nash, 1951). However, in a simultaneous-move game it is difficult to imagine that players respond to the opponents' actual strategies. Instead, it seems more natural to assume that they form beliefs about their opponents' strategies and then they respond to these beliefs. In this sense, one could say that a NE implicitly postulates that players respond optimally to their beliefs about their opponents'

*I am indebted to Amanda Friedenberg, Willemien Kets, Christian Seel, two anonymous referees and the audience in LOFT (Bergen) for useful comments and helpful discussions.

†Department of Economics (AE1), Maastricht University, P.O. Box 616, 6200 MD, Maastricht, The Netherlands; Tel: +31-43-38 83649; Fax: +31-43-38 84878; E-mail: e.tsakas@maastrichtuniversity.nl

strategies and these beliefs turn out to be correct. This alternative statement/interpretation of NE in terms of beliefs has inspired the literature on the epistemic foundations for NE, e.g., in their leading paper Aumann and Brandenburger (1995, p. 1161) begin with the preliminary observation that "*if each player is rational and knows the strategy choices of the others, the players' choices constitute a Nash equilibrium in the game being played*".

Formally, a belief is modeled with a probability measure over the set of the opponents' strategy profiles. Note that this does not necessarily need to be a product measure, implying that the player may in principle hold correlated beliefs, even if in reality her opponents choose their strategies independently from each other. A player may hold correlated beliefs, for instance, either because she thinks that her opponents use a correlating device (Aumann, 1974, 1987) or because she thinks that their opponents' beliefs are themselves correlated (Brandenburger and Friedenberg, 2008). Thus, as long as the players actually choose their strategies independently from each other, the notion of correct beliefs that is implicitly present in the definition of NE, requires each player (i) to have correct marginal beliefs about each individual opponent, and (ii) to have independent marginal beliefs.

In this paper, we drop the requirement that players have independent marginal beliefs, while maintaining the assumption that their marginal beliefs about each individual opponent are correct. This induces a new solution concept called correlated-belief equilibrium (CBE). Obviously, the predictions made by CBE are a coarsening of the corresponding NE predictions. Moreover, it is straightforward that the two solution concepts yield exactly the same strategy profiles in two-player games.

Our first aim is to provide an epistemic justification for this solution concept. To do so, we begin by looking into the history of the epistemic foundations for NE. In their seminal article, Aumann and Brandenburger (1995) proved that mutual belief in rationality and common belief in conjectures suffice for a NE if there is a common prior. This last assumption, while being often present in many game-theoretic results, has been extensively criticized mostly on the basis of its conceptual foundations being questionable (e.g., see Gul, 1998). In fact, Feinberg (2000) characterized the common prior assumption by means of a no-bet condition. In particular, he showed that a common prior exists if and only if the players cannot agree on any (zero-sum) bet. The fact that a common prior is characterized in terms of all possible bets makes it a rather strong assumption. This is because some bets are defined in terms of conditions that involve higher order beliefs, and in this respect they may not even be verifiable based on hard evidence, e.g., consider the bet according to which Bob has to pay Ann one monetary unit if she thinks that Carol will choose the strategy L with probability more than $1/2$, and she has to pay him one monetary unit otherwise. Obviously, in this case there is no way to verify whether Ann reports her true beliefs or not, when she says that she does indeed find it more likely that Carol will choose L.

Having recognized how restrictive the common prior assumption is, Barelli (2009) replaced it with a weaker condition, called action-consistency (A-consistency), which is characterized by the players not agreeing to take any bet *described in terms of the language generated by the pure strategy profiles*. In other words, A-consistency postulates that the players cannot agree on any (zero-sum) bet whose outcome depends solely on which pure strategy profile is played in the game. Note that in this case in order to decide who wins the side bet it is necessary to observe the realized pure strategy profile.

Let us now assume instead that the players cannot agree on any bet *described in terms of the language generated by any player's individual pure strategies*. That is, we postulate that the players cannot agree on any (zero-sum) bet whose outcome depends solely on an arbitrary single player's pure strategy. Let us call this no-bet condition I-consistency. Obviously, I-consistency is weaker than Barelli's A-consistency, as it requires players not being able to agree on fewer bets than A-consistency requires. Then, we show that by replacing A-consistency with I-consistency in Barelli's set of epistemic conditions (for NE) we provide sufficient epistemic conditions for CBE (see Theorem 1). We should point out that our epistemic conditions do not in general suffice for NE (see Example 2).

The previously-mentioned result justifies CBE by simply allowing for more priors than the ones that would lead to a NE. In this sense, similarly to the literature on the epistemic conditions for NE, it describes a set of belief hierarchies which would be consistent with CBE. Still, it does not explain how players would end up forming such beliefs, and consequently why they would end up behaving in accordance to the profiles that CBE predicts. We do this, by studying the relationship of CBE with other solution concepts, and in particular with conjectural equilibrium, a solution concept with well-established learning foundations which also permits false beliefs (e.g., see Hahn, 1977, 1978; Battigalli, 1987; Battigalli and Guaitoli, 1997).

Conjectural equilibrium does not necessarily require each player's belief to be correct in the probability it attaches to each pure strategy profile of the opponents. Instead, it assumes that each player receives a signal – upon each pure strategy profile being realized – and it requires each player's belief not to contradict the empirical distribution of the received signals. If the signals are precise enough to reveal the strategy profile being played, then conjectural equilibrium coincides with NE. Now, notice that observing such precise signals would lead each player to learn not only each opponent's mixed strategy distribution, but also the fact that the opponents choose independently from each other. On the other hand, in a CBE players are implicitly assumed to have learned each opponent's mixed strategy, without having necessarily learned that the opponents choose independently from each other. Indeed, we show that CBE can be rewritten as a new variant of conjectural equilibrium with multiple signals, each of them revealing one opponent's strategy (see Theorem 2). We also show that CBE is not a special case of the standard conjectural equilibrium with a single signal function for each player (see Example 3).

Besides the conceptual foundations of CBE, we relate the predictions that our concept makes with other standard solution concepts. Starting with rationalizability, we prove that every CBE is correlated rationalizable, i.e., it survives iterated elimination of strictly dominated strategies. This is rather straightforward to see, as the correct-marginal-beliefs assumption that we maintain, together with rationality, directly imply that the support of a CBE is a (correlated) best response set. On the other hand, this is not the case for independent rationalizability. In fact, it turns out that CBE and independent rationalizability neither refine nor coarsen each other. This is also natural to expect, as the two generalize NE in different directions. Now, turning to correlated equilibrium, we show that every CBE is essentially equivalent to a subjective correlated equilibrium. While we provide a constructive proof of this result which clearly illustrates the relationship between the two, notice that this also follows from the standard result of Brandenburger and Dekel (1987), who showed that every correlated rationalizable strategy profile is essentially equivalent to a refinement of subjective correlated equilibrium, viz., a posteriori equilibrium. Finally, we show that the same equivalence does not hold for objective correlated equilibrium.

The paper is structured as follows: Section 2 introduces some preliminary definitions, in Section 3 we define CBE and we prove some basic properties, Section 4 contains our epistemic foundation of the concept, while in Section 5 we provide the formal link to other solution concepts. All the proofs are relegated to the Appendix.

2. Preliminaries

2.1. Product measures

For a measurable space Y, let $\Delta(Y)$ denote the space of all probability measures over Y. If Y is finite, then $\Delta(Y)$ is identified as usual by the simplex over Y. Let $\mathrm{Supp}(\nu)$ denote the support of an arbitrary $\nu \in \Delta(Y)$. Now, take a finite collection of probability spaces $(Y_j, \nu_j)_{j=1}^n$, and denote the respective product measure over $\bigtimes_{j=1}^n Y_j$ by $\bigotimes_{j=1}^n \nu_j$. Moreover, let $\Pi\big(\bigtimes_{j=1}^n Y_j\big) \subseteq \Delta\big(\bigtimes_{j=1}^n Y_j\big)$ denote the set of all product (probability) measures over $\bigtimes_{j=1}^n Y_j$.

2.2. Normal form games

Consider a finite normal form game $\big(I, (A_i)_{i\in I}, (u_i)_{i\in I}\big)$, where $I = \{1, \dots, n\}$ denotes the finite set of players and A_i denotes the finite set of pure strategies of an arbitrary player $i \in I$ with typical element a_i. As usual, let $A := \bigtimes_{i\in I} A_i$ and $A_{-i} := \bigtimes_{j\neq i} A_j$, with typical elements $a = (a_1, \dots, a_n)$ and $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ respectively. Moreover, let $u_i : A \to \mathbb{R}$ denote player i's utility function.

A randomization over a player's pure strategies is called (mixed) strategy. Let $\Sigma_i := \Delta(A_i)$ denote the set of player i's mixed strategies with typical element σ_i. As usual, $\sigma_i(a_i)$ is the probability that σ_i attaches to a_i. Furthermore, let $\Sigma := \bigtimes_{i\in I} \Sigma_i$ denote the set of mixed strategy profiles with typical element $(\sigma_1, \dots, \sigma_n)$. As long as players are assumed to choose their strategies independently, we identify the mixed strategy profile $(\sigma_1, \dots, \sigma_n) \in \Sigma$ with the product measure $\sigma := \bigotimes_{i\in I} \sigma_i$. Likewise, let $\Sigma_{-i} := \bigtimes_{j\neq i} \Sigma_j$ denote the set of the strategy profiles chosen by i's opponents with typical element $(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$. Then again, as long as i's opponents choose their strategies independently, we identify $(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n) \in \Sigma_{-i}$ with the product measure $\sigma_{-i} := \bigotimes_{j\neq i} \sigma_j$.

Define i's (objective) expected utility from the mixed strategy profile $(\sigma_1, \dots, \sigma_n)$ by

$$\begin{aligned} U_i(\sigma_i, \sigma_{-i}) &:= \sum_{a_i\in A_i} \sigma_i(a_i) \sum_{a_{-i}\in A_{-i}} \sigma_{-i}(a_{-i}) \cdot u_i(a_i, a_{-i}) \\ &= \sum_{a\in A} \sigma(a) \cdot u_i(a). \end{aligned} \tag{1}$$

Then, we say that σ_i is a best response to $(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$ and we write $\sigma_i \in BR_i(\sigma_{-i})$ whenever it is the case that $U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Sigma_i$. Finally, recall that a strategy profile $(\sigma_1, \dots, \sigma_n)$ is a ***Nash Equilibrium (NE)*** whenever $\sigma_i \in BR_i(\sigma_{-i})$ for all $i \in I$.

2.3. Beliefs

A belief – often called conjecture – of player $i \in I$ is a probability measure $\mu_i \in \Delta(A_{-i})$. Notice that μ_i is not necessarily a product measure over A_{-i}, thus implying that player i may believe that her opponents' strategies are correlated, even if in reality they are chosen independently. The following definition formalizes what it means for a player to have independent (marginal) beliefs.

Definition 1. Player i has ***independent (marginal) beliefs*** whenever $\mu_i \in \Pi(A_{-i})$. On the other hand, we say that player i has ***correlated beliefs*** whenever $\mu_i \in \Delta(A_{-i}) \setminus \Pi(A_{-i})$.

While we allow players to have correlated beliefs, we do not impose any assumption regarding the source of correlation. That is, player i may hold correlated beliefs either because she believes that her opponents j and k actually use a physical correlating device similarly to Aumann (1974, 1987), or because she believes that j and k have themselves correlated belief hierarchies like for instance in Brandenburger and Friedenberg (2008). The former type of correlation is called extrinsic correlation, whereas the latter is called intrinsic correlation.

We define i's (subjective) expected utility from the mixed strategy $\sigma_i \in \Sigma_i$ given the conjecture $\mu_i \in \Delta(A_{-i})$ by

$$\begin{aligned} U_i(\sigma_i, \mu_i) &:= \sum_{a_i \in A_i} \sigma_i(a_i) \sum_{a_{-i} \in A_{-i}} \mu_i(a_{-i}) \cdot u_i(a_i, a_{-i}) \\ &= \sum_{a \in A} (\sigma_i \otimes \mu_i)(a) \cdot u_i(a). \end{aligned} \tag{2}$$

As usual, we say that $\sigma_i \in \Sigma_i$ is a rational/optimal strategy given the belief $\mu_i \in \Delta(A_{-i})$, and we write $\sigma_i \in BR_i(\mu_i)$, whenever it is the case that $U_i(\sigma_i, \mu_i) \geq U_i(\sigma_i', \mu_i)$ for all $\sigma_i' \in \Sigma_i$. Then, it is straightforward to verify that a mixed strategy profile $(\sigma_1, \ldots, \sigma_n)$ is a NE if and only if for every $i \in I$ there is a conjecture $\mu_i \in \Delta(A_{-i})$ such that $\sigma_i \in BR_i(\mu_i)$ and $\mu_i = \sigma_{-i}$. Whenever it is the case that $\mu_i = \sigma_{-i}$ we say that player i has correct beliefs.

3. Definition and basic properties

The correct beliefs assumption, which is implicitly present in the definition of NE, essentially postulates that each player (i) correctly guesses each opponent's mixed strategy, and (ii) has independent beliefs. Formally, player i has correct beliefs whenever μ_i satisfies the following two conditions:

Correct marginal beliefs (CMB): $\operatorname{marg}_{A_j} \mu_i = \sigma_j$ for all $j \neq i$.

Independent (marginal) beliefs (IB): $\mu_i \in \Pi(A_{-i})$.

In this paper, we partially relax the correct beliefs assumption, by dropping IB while maintaining CMB. That is, while we still assume that each player has a correct marginal belief about each individual opponent, we do not require her to have independent marginal beliefs. As a result we obtain a new solution concept, which we call correlated-belief equilibrium.

Definition 2. A strategy profile $(\sigma_1, \ldots, \sigma_n)$ is a ***correlated-belief equilibrium (CBE)*** whenever there exists a tuple of conjectures $(\mu_1, \ldots, \mu_n)$ such that for all $i \in I$,

(*a*) $\sigma_i \in BR_i(\mu_i)$,

(*b*) $\operatorname{marg}_{A_j} \mu_i = \sigma_j$ for all $j \neq i$.

It is obvious that the set of CBE is a coarsening of the set of NE. The following result formally proves that this is the case.

Proposition 1. *Every NE is a CBE.*

Though rather straightforward, it is worthwhile to mention that the existence of a CBE follows directly from the previous result.

Corollary 1. *Every finite normal form game has a CBE.*

In general the converse of Proposition 1 is not true, as illustrated by the following example. In particular, it turns out that CBE may predict strictly more strategy profiles than NE does.

Example 1. Consider the following three-player game, with $I = \{\text{Ann } (a), \text{ Bob } (b), \text{ Carol } (c)\}$. Ann chooses the matrix, Bob the row and Carol the column, i.e., $A_a = \{L, R\}$, $A_b = \{A, B\}$ and $A_c = \{C, D\}$. Furthermore, the payoffs are written in the respective order, i.e., first Ann, then Bob and then Carol. Now consider the mixed strategy profile $(\sigma_a, \sigma_b, \sigma_c)$ where $\sigma_a = (1 \circledast L)$,

	L		R	
	C	D	C	D
A	1,1,1	0,0,0	1,2,2	1,2,2
B	0,0,0	1,1,1	1,2,2	1,2,2

$\sigma_b = (\frac{1}{2} \circledast A\ ;\ \frac{1}{2} \circledast B)$ and $\sigma_c = (\frac{1}{2} \circledast C\ ;\ \frac{1}{2} \circledast D)$. Note that this is not a NE, as Ann has an incentive to deviate to $(1 \circledast R)$, which would yield expected payoff equal to 1, instead of the 1/2 that σ_a yields. In other words, σ_a is not a rational strategy, if Ann has independent beliefs. However, σ_a can be sustained as a rational strategy if her conjecture is $\mu_a = \big(\frac{1}{2} \circledast (A, C)\ ;\ \frac{1}{2} \circledast (B, D)\big)$. In this case, Ann has correct marginal beliefs, but not independent beliefs, viz., while μ_a is such that $\text{marg}_{A_i}\, \mu_a = \sigma_i$ for each $i \in \{b, c\}$, it is not the case $\mu_a \notin \Pi(A_b \times A_c)$. Finally, notice that $\sigma_i \in BR_i(\sigma_{-i})$ for every $i \in \{b, c\}$, thus implying that $(\sigma_a, \sigma_b, \sigma_c)$ is a CBE. ◁

Still, some partial converse results can be established. First, we show that in two-player games the set of NE coincides with the set of CBE. This is not surprising, as the only difference between a CBE and a NE is that the latter requires the players to have independent beliefs. However, if there are only two players in the game, each of them has a unique opponent and therefore the marginal beliefs are trivially independent.

Proposition 2. *In two-player games every CBE is a NE.*

Second, we show that the set of pure strategy NE coincides with the set of pure strategy CBE. Again, the reason is rather obvious, viz., if every player chooses a pure strategy, the only conjecture with correct marginal beliefs is equal to the product measure of the opponents' (pure) strategies.

Proposition 3. *Every pure-strategy CBE is a NE.*

4. Epistemic foundations

In this section, we provide sufficient epistemic conditions for CBE, thus obtaining a foundation for CBE. In particular, our conditions weaken the standard epistemic foundations for NE by Aumann and Brandenburger (1995) and Barelli (2009).

4.1. Epistemic models

A belief hierarchy describes a player's belief about the opponents' strategies (first order beliefs), belief about the opponents' strategies and first order beliefs (second order beliefs), and so on. Formally, for an arbitrary player i consider the following sequence of (Polish) spaces:[1] For each player $i \in I$, let $\Theta_i^0 := A_{-i}$, and for each $k > 0$ recursively define $\Theta_i^k := \Theta_i^{k-1} \times \left(\times_{j \neq i} \Delta(\Theta_j^{k-1})\right)$. Then, a belief hierarchy of player i is a sequence of Borel probability measures $(\mu_i^1, \mu_i^2, \dots) \in \times_{k \geq 0} \Delta(\Theta_i^k)$ satisfying coherency and common certainty in coherency, with $H_i \subseteq \times_{k \geq 0} \Delta(\Theta_i^k)$ being the set of all such belief hierarchies.[2] As usual, $\mu_i^k \in \Delta(\Theta_i^{k-1})$ denotes the k-th order beliefs, and $\mu_i^1 \in \Delta(A_{-i})$ coincides with i's conjecture.

Belief hierarchies are typically represented using type space models. A type space model is a tuple $\left((T_i)_{i \in I}, (\lambda_i)_{i \in I}\right)$, where T_i is a Polish space of player i's types, and $\lambda_i : T_i \to \Delta(A_{-i} \times T_{-i})$ is a continuous function, with $T_{-i} := \times_{j \neq i} T_j$. Throughout the paper we consider countable type space models. This is without loss of generality, as our analysis can be directly generalized to arbitrary type space models. Each type $t_i \in T_i$ is associated with a belief hierarchy $h_i(t_i) := \left(\mu_i^1(t_i), \mu_i^2(t_i), \dots\right) \in H_i$, where the k-th order beliefs assign probability

$$\mu_i^k(t_i)(E) := \int_{(a_{-i}, t_{-i}) \in A_{-i} \times T_{-i} \,:\, (a_j, \mu_j^1(t_j), \dots, \mu_j^{k-1}(t_j))_{j \neq i} \in E} d\lambda_i(t_i) \tag{3}$$

to an arbitrary Borel subset $E \subseteq \Theta_i^0 \times \left(\times_{j \neq i} \mathrm{Proj}_{\Delta(\Theta_j^0) \times \cdots \times \Delta(\Theta_j^{k-2})} H_j\right)$.

For a given type space model, $S := \times_{i \in I}(A_i \times T_i)$ is the set of states (of the world) with typical element s. Each Borel subset of S is called an event. Note that a state specifies each player's pure strategy as well as her entire belief hierarchy. In particular, for each $i \in I$, there

[1] Recall that a space is Polish if it is separable and completely metrizable. Recall that the countable product of Polish spaces is also Polish. Moreover, if Y is Polish, so is $\Delta(Y)$ endowed with the topology of weak convergence. For further details on Polish spaces, we refer to Aliprantis and Border (1994).

[2] Recall that a belief hierarchy $(\mu_i^1, \mu_i^2, \dots)$ is coherent whenever it is the case that $\mathrm{marg}_{\Theta_{k-2}} \mu_i^k = \mu_i^{k-1}$ for all $k > 1$, and we denote the space of i's coherent belief hierarchies by H_i^1. Then, for each $\ell > 0$ we recursively define $H_i^\ell := \left\{ (\mu_i^1, \mu_i^2, \dots) \in H_i^1 \,:\, \mu_i^{k+2}\left(\Theta_i^0 \times \left(\times_{j \neq i} \mathrm{Proj}_{\Delta(\Theta_j^0) \times \cdots \times \Delta(\Theta_j^k)} H_j^{\ell-1}\right)\right) = 1 \text{ for all } k \geq 0 \right\}$ as the set of belief hierarchies satisfying ℓ-fold certainty in coherency, and we say that a belief hierarchy satisfies common certainty in coherency if it is an element of $H_i := \bigcap_{\ell \geq 1} H_i^\ell$ (Harsanyi, 1967-68; Mertens and Zamir, 1985; Brandenburger and Dekel, 1993).

exists a function $\tilde{a}_i : S \to A_i$, defined by $\tilde{a}_i(s) := \text{Proj}_{A_i}\{s\}$ for an arbitrary $s \in S$. Then, let $[a_i] := \{s \in S : \tilde{a}_i(s) = a_i\}$ denote the event that player i has chosen the pure strategy $a_i \in A_i$, and as usual define the events $[a] := \bigcap_{i \in I}[a_i]$ and $[a_{-i}] := \bigcap_{j \neq i}[a_j]$. Likewise, there exists a function $\tilde{t}_i : S \to T_i$, defined by $\tilde{t}_i(s) := \text{Proj}_{T_i}\{s\}$ for an arbitrary $s \in S$. Then, let $[t_i] := \{s \in S : \tilde{t}_i(s) = t_i\}$ denote the event that player i's type is $t_i \in T_i$. Now, each state is indirectly associated with a belief hierarchy, via the function $\tilde{\mu}_i^k := \mu_i^k \circ \tilde{t}_i$, i.e., at a state $s \in S$, player i's k-th order beliefs are given by $\tilde{\mu}_i^k(s) := \mu_i^k(\tilde{t}_i(s))$, while i's belief hierarchy at s is denoted by $\tilde{h}_i(s) := \big(\tilde{\mu}_i^1(s), \tilde{\mu}_i^2(s), \dots\big)$. Once again, i's conjecture coincides with the first order beliefs, i.e., at each $s \in S$, player i's conjecture is denoted by $\tilde{\mu}_i(s) := \tilde{\mu}_i^1(s)$. Then, let $[\mu_i] := \{s \in S : \tilde{\mu}_i(s) = \mu_i\}$ denote the event that player i's conjecture is μ_i. Finally, the event

$$R_i := \Big\{ \ s \in S : \tilde{a}_i(s) \in BR_i\big(\tilde{\mu}_i(s)\big) \ \Big\} \tag{4}$$

contains the states where player i is rational.

Now, for every $s \in S$ define the Borel probability measure $\tilde{\beta}_i(s) \in \Delta(S)$, such that for an arbitrary Borel event $E \subseteq S$,

$$\tilde{\beta}_i(s)(E) := \lambda_i\big(\tilde{t}_i(s)\big)\Big(\text{Proj}_{A_{-i} \times T_{-i}}\big(E \cap [\tilde{a}_i(s)] \cap [\tilde{t}_i(s)]\big)\Big) \tag{5}$$

denotes the probability that player i attaches to E at the state s. Note that player i is implicitly assumed to know both her own strategy and her own type at every state, which is why she attaches probability 0 to every event that contradicts her actual strategy-type profile.

We say that player i believes in E at all states that belong to the event

$$B_i(E) := \{s \in S : \tilde{\beta}_i(s)(E) = 1\}. \tag{6}$$

Moreover, we say that E is mutually believed if everybody believes in it, and we write

$$B(E) := \bigcap_{i \in I} B_i(E). \tag{7}$$

Finally, we say that E is commonly believed if everybody believes in it, everybody believes that everybody believes in it, and so on. Formally, for each $m \geq 1$ we recursively define m-th order mutual belief in E by $B^m(E) := B^{m-1}(B(E))$ where $B^0(E) := B(E)$. Then, the states that belong to the event

$$CB(E) := \bigcap_{m \geq 0} B^m(E) \tag{8}$$

are those at which E is commonly believed.

4.2. Epistemic conditions for Nash equilibrium

In their seminal paper, Aumann and Brandenburger (1995) provided a set of sufficient epistemic conditions for NE, by showing that in a complete information game, common belief of conjectures and mutual belief in rationality suffice for a NE, whenever there is a common prior. Recall that a Borel probability measure $q \in \Delta(S)$ is a ***common prior*** if for every player $i \in I$ and for every $s \in S$ with $q\big([\tilde{a}_i(s)] \cap [\tilde{t}_i(s)]\big) > 0$ it is the case that $\tilde{\beta}_i(s)(E) = q\big(E \cap [\tilde{a}_i(s)] \cap [\tilde{t}_i(s)]\big)/q\big([\tilde{a}_i(s)] \cap [\tilde{t}_i(s)]\big)$ for every Borel $E \subseteq S$. Then, Aumman and Brandenburger's result is formally stated as follows.[3]

Theorem A (Aumann and Brandenburger, 1995)**.** *Let $(\mu_1, \ldots, \mu_n)$ be a tuple of conjectures and suppose that there is a common prior $q \in \Delta(S)$ that attaches positive probability to a state $s \in S$ such that $s \in B(R_1 \cap \cdots \cap R_n) \cap CB\big([\mu_1] \cap \cdots \cap [\mu_n]\big)$. Then, there exists a mixed strategy profile $(\sigma_1, \ldots, \sigma_n) \in \Sigma$ such that*

(i) $\mathrm{marg}_{A_i} \mu_j = \sigma_i$ *for all $j \neq i$ and for all $i \in I$,*

(ii) $(\sigma_1, \ldots, \sigma_n)$ *is a NE.*

In a more recent paper, Barelli (2009) generalized Aumann and Brandenburger's epistemic conditions, by substituting the common prior assumption and common belief in conjectures with weaker conditions respectively. Formally, Barelli (2009) relaxed the common prior assumption, by introducing the weaker notion of action-consistency. Accordingly, consider the set of A-measurable random variables, $\mathcal{F}_A := \{f : S \to \mathbb{R} \mid \tilde{a}(s) = \tilde{a}(s') \Rightarrow f(s) = f(s')\}$. Henceforth, a function $f \in \mathcal{F}_A$ is called A-verifiable, as the value of f at some state reveals the strategy profile being played at this state. Then, a probability measure $q \in \Delta(S)$ is called ***action-consistent (A-consistent)*** whenever it is the case that

$$\sum_{s \in S} q(s) \cdot f(s) = \sum_{s \in S} q(s) \Big(\sum_{s' \in S} \tilde{\beta}_i(s)(s') \cdot f(s') \Big) \tag{9}$$

for every $i \in I$ and every $f \in \mathcal{F}_A$. In fact, whenever A is finite, Equation (9) implies

$$q\big([a]\big) = \sum_{s \in S} \tilde{\beta}_i(s)\big([a]\big) \cdot q(s) \tag{10}$$

for every $i \in I$ and every $a \in A$.

Then, Barelli (2009) characterized of A-consistency in terms of A-verifiable bets. We define a bet as a collection $\{f_i\}_{i \in I}$ of random variables such that $\sum_{i \in I} f_i(s) = 0$ for all $s \in S$. A bet is A-verifiable whenever $f_i \in \mathcal{F}_A$ for all $i \in I$. Then, it can be shown that there is an A-consistent

[3] In their paper, Aumann and Brandenburger (1995) prove a slightly more general version of this result, viz., they do not require common belief of the payoff functions, which we implicitly assume here.

probability measure in $\Delta(S)$ if and only if there is *no* mutually beneficial A-verifiable bet, i.e., formally, $q \in \Delta(S)$ is A-consistent if and only if there exists *no* A-verifiable bet $\{f_i\}_{i\in I}$ such that $\sum_{s'\in S} \tilde{\beta}_i(s)(s') \cdot f_i(s') \geq 0$ for all $s \in \mathrm{Supp}(q)$ and for all $i \in I$, with at least one inequality being strict. Furthermore, for some A-consistent measure $q \in \Delta(S)$, the conjectures are said to be constant in the support of q whenever there exists a profile of conjectures $(\mu_1, \ldots, \mu_n)$ such that $\big(\tilde{\mu}_1(s), \ldots, \tilde{\mu}_n(s)\big) = (\mu_1, \ldots, \mu_n)$ for all $s \in \mathrm{Supp}(q)$.

Then, Barelli (2009) generalized Aumann and Brandenburger's epistemic conditions for NE, by simultaneously replacing their common prior assumption with A-consistency, and common belief of the conjectures with constant conjectures in the support of the A-consistent distribution. Formally, Barelli's result is stated as follows.

Theorem B (Barelli, 2009)**.** *Let $(\mu_1, \ldots, \mu_n)$ be a tuple of conjectures and suppose that there is an A-consistent $q \in \Delta(S)$ such that $\big(\tilde{\mu}_1(s), \ldots, \tilde{\mu}_n(s)\big) = (\mu_1, \ldots, \mu_n)$ for all $s \in \mathrm{Supp}(q)$. Moreover, assume that there is some state $s \in \mathrm{Supp}(q)$ such that $s \in B(R_1 \cap \cdots \cap R_n)$. Then, there exists a mixed strategy profile $(\sigma_1, \ldots, \sigma_n) \in \Sigma$ such that*

(i) $\mathrm{marg}_{A_i} \mu_j = \sigma_i$ *for all $j \neq i$ and for all $i \in I$,*

(ii) $(\sigma_1, \ldots, \sigma_n)$ *is a NE.*

To see that Barelli's conditions are weaker than Aumann and Brandenburger's, first observe that a common prior is always A-consistent. This is not surprising given the existing characterizations of the two concepts. In particular, a common prior essentially says that the players will not agree to take any bet that is defined in terms of events in the Borel σ-algebra $\mathcal{S}$ of events in S (Feinberg, 2000), whereas A-consistency essentially says that the players will not agree to take any bet that is defined in terms of events in the sub-σ-algebra $\mathcal{S}_A$ which is generated by the collection $\{[a] | a \in A\}$ (Barelli, 2009). Obviously, the former is more restrictive than the latter – by the fact that $\mathcal{S}_A \subseteq \mathcal{S}$ – which is in accordance to our observation that a common prior is A-consistent.

Finally, in the existence of a common prior, constant conjectures in the support of the (action-consistent) common prior directly implies common belief of the conjectures.

4.3. Epistemic conditions for correlated-belief equilibrium

According to Barelli's characterization, an A-consistent probability measure exists if and only if there is no mutually beneficial bet that is described in terms of *all players' pure strategies*. Now, let us relax this condition by instead assuming that players are not able to agree on a bet which is described in terms of *a single player's pure strategies*. In order to do this, we first define, for an arbitrary $i \in I$, the set of A_i-measurable random variables, $\mathcal{F}_{A_i} := \{f : S \to \mathbb{R} \mid \tilde{a}_i(s) = \tilde{a}_i(s') \Rightarrow f(s) = f(s')\}$. Henceforth, a function $f \in \mathcal{F}_{A_i}$ is called A_i-verifiable, as the value of f at some state reveals the pure strategy chosen *by player* i at this state.

Definition 3. A probability measure $q \in \Delta(S)$ is called A_i***-consistent*** whenever it is the case that

$$\sum_{s \in S} q(s) \cdot f(s) = \sum_{s \in S} q(s) \Big(\sum_{s' \in S} \tilde{\beta}_j(s)(s') \cdot f(s') \Big) \tag{11}$$

for every $j \in I$ and every $f \in \mathcal{F}_{A_i}$. Moreover, we say that $q \in \Delta(S)$ is I***-consistent***, whenever it is A_i-consistent for every $i \in I$.

Similarly to Barelli (2009), it can be shown that whenever A_i is finite, Equation (11) implies

$$q\big([a_i]\big) = \sum_{s \in S} \tilde{\beta}_j(s)\big([a_i]\big) \cdot q(s) \tag{12}$$

for every $j \in I$ and every $a_i \in A_i$. Hence, if q is I-consistent, (12) holds for every $i \in I$. Then, following the same steps as in Barelli (2009, Prop. 5.3), it can be shown that there is an A_i-consistent probability measure in $\Delta(S)$ if and only if there is *no* mutually beneficial A_i-verifiable bet, viz., $q \in \Delta(S)$ is A_i-consistent if and only if there exists *no* A_i-verifiable bet $\{f_j\}_{j \in I}$ such that

$$\sum_{s' \in S} \tilde{\beta}_j(s)(s') \cdot f_j(s') \geq 0 \tag{13}$$

for all $s \in \mathrm{Supp}(q)$ and for all $j \in I$, with at least one inequality being strict. Thus, $q \in \Delta(S)$ is I-consistent if and only if there is no A_i-verifiable bet satisfying (13) for any $i \in I$.

The intuition behind the previous characterization of A_i-consistency is the players will not agree to take any bet that is defined in terms of events in the sub-σ-algebra $\mathcal{S}_{A_i}$ which is generated by the collection $\{[a_i] | a_i \in A_i\}$. That is, I-consistency essentially says that the players will not agree on any bet that is described in terms of events in $\mathcal{S}_{A_1}$, and they will not agree on any bet that is described in terms of events in $\mathcal{S}_{A_2}$, and $\ldots$, and and they will not agree on any bet that is described in terms of events in $\mathcal{S}_{A_n}$. Notice that for every $i \in I$ it is the case that $\mathcal{S}_{A_i} \subseteq \mathcal{S}_A$, thus implying that every A-consistent prior is also I-consistent.

Then, the following result proves that if we replace A-consistency with I-consistency in Barelli's Theorem, the marginal conjectures will form a CBE rather than a NE. Thus, we provide sufficient epistemic conditions for CBE.

Theorem 1. *Let* $(\mu_1, \ldots, \mu_n)$ *be a tuple of conjectures and suppose that there is an* I*-consistent* $q \in \Delta(S)$ *such that* $\big(\tilde{\mu}_1(s), \ldots, \tilde{\mu}_n(s)\big) = (\mu_1, \ldots, \mu_n)$ *for all* $s \in \mathrm{Supp}(q)$*. Moreover, assume that there is some state* $s \in \mathrm{Supp}(q)$ *such that* $s \in B(R_1 \cap \cdots \cap R_n)$*. Then, there exists a mixed strategy profile* $(\sigma_1, \ldots, \sigma_n) \in \Sigma$ *such that* *(i)* $\text{marg}_{A_i} \mu_j = \sigma_i$ *for all* $j \neq i$ *and for all* $i \in I$,

(ii) $(\sigma_1, \ldots, \sigma_n)$ *is a CBE.*

Note that I-consistency does not force players to agree on the probabilities they attach an arbitrary event in $\mathcal{S}_A$, e.g., Ann and Bob do not need to have the same conjecture about Carol and David jointly. However, it is still the case that under I-consistency all players other than i necessarily agree on the probabilities they attach to each event in $\mathcal{S}_{A_i}$, e.g., Ann and Bob necessarily have the same marginal conjecture about Carol, as well as the same marginal conjecture about David. This, explains why in a CBE players can have different conjectures, but not different marginal conjectures. On the other hand, under A-consistency, players agree not only on their marginal beliefs, but also on their joint beliefs – as everybody's conjectures are product measures – and therefore A-consistency suffices for NE.

Still, it is natural to ask whether our (weaker) conditions of Theorem 1 also suffice for NE.[4] It turns out that this is not the case, as shown in the following example.

Example 2. Recall the game in Example 1, and consider the CBE $\sigma_a = (1 \circledast L)$, $\sigma_b = (\frac{1}{2} \circledast A\ ;\ \frac{1}{2} \circledast B)$ and $\sigma_c = (\frac{1}{2} \circledast C\ ;\ \frac{1}{2} \circledast D)$. Now, consider a type space model $\big((T_i)_{i \in I}, (\lambda_i)_{i \in I}\big)$ with a unique type for each player. i.e., $T_i = \{t_i\}$ for each $i \in \{a, b, c\}$, where

$$\begin{aligned}
\lambda_a(t_a) &= \Big(\frac{1}{2} \circledast \big((A, t_b), (C, t_c)\big)\ ;\ \frac{1}{2} \circledast \big((B, t_b), (D, t_c)\big)\Big), \\
\lambda_b(t_b) &= \Big(\frac{1}{2} \circledast \big((L, t_a), (C, t_c)\big)\ ;\ \frac{1}{2} \circledast \big((L, t_a), (D, t_c)\big)\Big), \\
\lambda_c(t_c) &= \Big(\frac{1}{2} \circledast \big((L, t_a), (A, t_b)\big)\ ;\ \frac{1}{2} \circledast \big((L, t_a), (B, t_b)\big)\Big).
\end{aligned}$$

Moreover, consider the probability measure $q \in \Delta(S)$ defined by

$$q = \Big(\frac{1}{2} \circledast \big((L, t_a), (A, t_b), (C, t_c)\big)\ ;\ \frac{1}{2} \circledast \big((L, t_a), (B, t_b), (D, t_c)\big)\Big)$$

and notice that it is I-consistent, while at the same time the conjectures are constant in the support of q. Thus, the conditions of Theorem 1 are satisfied, and as expected $(\sigma_a, \sigma_b, \sigma_c)$ is a CBE. However, recall from Example 1 that $(\sigma_a, \sigma_b, \sigma_c)$ is not a NE, thus implying that the conditions of Theorem 1 do not suffice for NE. Of course, the reason is that q is not A-consistent. Indeed, consider the A-measurable (indicator) function $f : S \to \mathbb{R}$ with $f(s) = 1$ for all $s \in \text{Supp}(q)$ and $f(s) = 0$ otherwise, and notice that while $\sum_{s \in S} q(s) \cdot f(s) = 1$, it is also the case that $\sum_{s \in S} q(s) \big(\sum_{s' \in S} \tilde{\beta}_i(s)(s')\big) = 1/2$ for $i \in \{b, c\}$, thus implying that Equation (9) does not hold for every player. $\triangleleft$

[4] Recall that both Aumann and Brandenburger (1995), as well as the subsequent generalizations including Barelli (2009), provide sufficient but not necessary conditions for NE.

Remark 1. Obviously, in two player games, it suffices to simply require mutual belief in conjectures similarly to Aumann and Brandenburger (1995, Thm. A). The reason is that – as we have already mentioned – in two player games the set of CBE coincides with the set of NE. $\triangleleft$

Remark 2. In a recent paper, Bach and Tsakas (2014) further generalized Barelli's result – and a fortiori the one of Aumann and Brandenburger (1995) – by replacing his epistemic conditions with respective pairwise epistemic conditions imposed only for some pairs of players. A similar generalizations can be done for CBE, by assuming pairwise mutual belief in rationality, pairwise I-consistency and pairwise constant conjectures in the supports of the I-consistent distributions. However, due to space limitations, we omit the presentation of this result. $\triangleleft$

Remark 3. Notice that throughout this section, for notation simplicity, we have focused entirely on complete information games, even though our result can be directly extended to type spaces with uncertainty about the payoff functions, as long as these are mutually believed. $\triangleleft$

5. Relationship to other solution concepts

5.1. Conjectural equilibrium

CBE is not the first equilibrium concept in the literature that allows for correlated beliefs. In fact, this idea was already present in conjectural equilibrium, which was introduced by Hahn (1977, 1978), later formalized by Battigalli (1987) and further developed by Battigalli and Guaitoli (1997) and Gilli (1999).[5] The underlying idea behind all these papers is that a player's beliefs do not need to coincide with the product measure induced by the opponents' actual mixed strategy profile. Instead, beliefs should only be confirmed for some events in A_{-i}.

Formally, for an arbitrary player $i \in I$, let $\psi_i : A \to M_i$ be a signal function with M_i being an arbitrary set of signals. Whenever the pure strategy profile $a \in A$ is played, player i does not necessarily observe it – even after the strategies have been realized – and instead receives the signal $\psi_i(a)$, thus considering possible the pure strategy profiles in

$$\psi_i^{-1}(m_i) := \{a \in A : \psi_i(a) = m_i\}. \tag{14}$$

A natural example of a signal function is the own utility function, viz., suppose that player i observes only his own utility, but not the pure strategy profile that induced it. Then, for an arbitrary $a_i \in A_i$,

[5] This literature consists of a whole family of related solution concepts, such as for instance rationalizable conjectural equilibrium (Rubinstein and Wolinsky, 1994; Esponda, 2013), self-confirming equilibrium (Fudenberg and Levine, 1993) and subjective equilibrium (Kalai and Lehrer, 1993), just to mention a few.

let

$$\psi_i^{-1}(m_i|a_i) := \{a_{-i} \in A_{-i} : \psi_i(a_i, a_{-i}) = m_i\} \tag{15}$$

be the set of the opponents' pure strategy profiles which together with a_i would induce the signal m_i. A conjectural equilibrium captures the idea each player responds rationally to a (possibly wrong) conjecture about the strategies chosen by their opponents. Still, this conjecture – though possibly wrong – is not contradicted by the signal that the player observes after the actions have been realized. Formally, for a profile of signal functions $\psi = (\psi_1, \ldots, \psi_n)$, we say that a strategy profile $(\sigma_1, \ldots, \sigma_n)$ is a ***ψ-conjectural equilibrium***, if for every $i \in I$ there is a belief $\mu_i \in \Delta(A_{-i})$ such that

(*a*) $\mu_i\big(\psi_i^{-1}(m_i|a_i)\big) = \sigma_{-i}\big(\psi_i^{-1}(m_i|a_i)\big)$ for all $m_i \in M_i$ and for all $a_i \in \mathrm{Supp}(\sigma_i)$,

(*b*) $\sigma_i \in BR_i(\mu_i)$.

The underlying idea is that an arbitrary player $i \in I$ observes a large sample of realized combinations of the own action and the corresponding signal, $\big(a_i, \psi_i(a)\big)$. Then, the probability that the conjecture μ_i assigns to each signal (conditional on each own pure strategy) is equal to the empirical frequency of this signal (conditional on each own pure strategy). Therefore, i's conjecture is confirmed by the observed sample.

If the signal function of each player induces the discrete partition over A, i.e., if $\psi_i(a) \neq \psi_i(a')$ for any $a, a' \in A$, a strategy profile is a conjectural equilibrium if and only if it is a NE. Moreover, it is straightforward that for any given signal functions $(\psi_1, \ldots, \psi_n)$ the set of conjectural equilibria lies between NE and correlated rationalizability, similarly to what happens with CBE.[6] The natural question arising then is whether there exists some ψ such that the set of CBE coincides with the set of ψ-conjectural equilibria. The following example shows that this is not necessarily the case, thus implying that CBE is not a special case of conjectural equilibrium.

Example 3. Consider the following three-player game, with $I = \{\text{Ann } (a), \text{ Bob } (b), \text{ Carol } (c)\}$. Ann chooses the matrix, Bob the row and Carol the column, i.e., $A_a = \{L, R\}$, $A_b = \{A, B\}$ and $A_c = \{C, D\}$. Furthermore, the payoffs are written in the respective order, i.e., first Ann, then Bob and then Carol. Now take the mixed strategy profile $(\sigma_a, \sigma_b, \sigma_c)$ where $\sigma_a = (1 \circledast L)$, $\sigma_b = (\frac{1}{2} \circledast A\ ;\ \frac{1}{2} \circledast B)$ and $\sigma_c = (\frac{1}{2} \circledast C\ ;\ \frac{1}{2} \circledast D)$, and observe that it is a CBE. In fact, it is the only CBE with Ann choosing L. Then, let us show that there is no $\psi = (\psi_a, \psi_b, \psi_c)$ such that the set of ψ-conjectural equilibria coincide with the set of CBE. In particular, we show that for every ψ, either there is a ψ-conjectural equilibrium that is not a CBE, or it is the case that $(\sigma_a, \sigma_b, \sigma_c)$ is not a ψ-conjectural equilibrium. First, observe that in order for $(\sigma_a, \sigma_b, \sigma_c)$ to be a ψ-conjectural equilibrium

[6] The latter is formally proven in the next section.

	L		R	
	C	D	C	D
A	2,0,0	0,1,1	2,0,0	2,1,1
B	0,1,1	2,0,0	2,1,1	2,0,0

it must be the case that $\text{Supp}(\mu_a) \subseteq \{(A, C), (B, D)\}$. Moreover, notice that ψ_a must be such that $\psi_a(L, A, D) = \psi_a(L, A, C)$ or $\psi_a(L, A, D) = \psi_a(L, B, D)$, and likewise $\psi_a(L, B, C) = \psi_a(L, A, C)$ or $\psi_a(L, B, C) = \psi_a(L, B, D)$. Otherwise, $(\sigma_a, \sigma_b, \sigma_c)$ will not be a ψ-conjectural equilibrium. This is because, $\sigma_{-a}(A, D) > 0$ whereas $\mu_a(A, D) = 0$, and likewise $\sigma_{-a}(B, C) > 0$ whereas $\mu_a(B, C) = 0$. Let us assume that $\psi_a(L, A, D) = \psi_a(L, A, C)$. This is without loss of generality due to symmetry. Then, irrespective of what ψ_b and ψ_c are, it is the case that (L, A, D) is a ψ-conjectural equilibrium but not a CBE. Indeed, verify that this is the case by taking $\mu'_a = (1 \circledast (A, C))$, $\mu'_b = (1 \circledast (L, D))$ and $\mu'_c = (1 \circledast (L, A))$. Hence, we conclude that there is no signal structure ψ that makes the set of ψ-conjectural equilibria coincide with the set of CBE. $\triangleleft$

While CBE is not a special case of conjectural equilibrium, there still seems to exist a very close link between the two. Indeed, as we show below, a CBE is indeed equivalent to a new variant of *conjectural equilibrium with multiple signals.*

Let us begin by defining for each player $i \in I$ the collection $\Psi_i := \{\psi_i^1, \dots, \psi_i^{i-1}, \psi_i^{i+1}, \dots, \psi_i^n\}$ of signal functions such that for an arbitrary $j \in I$ the signal function $\psi_i^j : A \to M_i$ reveals j's pure strategy, i.e., $\psi_i^j(a_1, \dots, a_n) = \psi_i^j(a'_1, \dots, a'_n)$ if and only if $a_j = a'_j$. Then, we say that $(\sigma_1, \dots, \sigma_n)$ is a Ψ-***conjectural equilibrium*** if for every $i \in I$ there is a belief $\mu_i \in \Delta(A_{-i})$ such that

(*a*) $\mu_i\big(\psi_i^{-1}(m_i|a_i)\big) = \sigma_{-i}\big(\psi_i^{-1}(m_i|a_i)\big)$ for all $\psi_i \in \Psi_i$ and all $m_i \in M_i$ and all $a_i \in \text{Supp}(\sigma_i)$,

(*b*) $\sigma_i \in BR_i(\mu_i)$.

Theorem 2. *A strategy profile is a CBE if and only if it is a Ψ-conjectural equilibrium.*

The previous result provides a natural characterization of CBE. Indeed, recall that a conjectural equilibrium allows the players' conjectures to be false, but still requires them to be consistent with the observed data which arrive in the form of signals. Thus, a Ψ-conjectural equilibrium postulates that each players receives separate data for each opponent's strategy and confirms her marginal conjectures. However, she is not able – using these independently received signals – to test whether her opponents' strategies are statistically independent or not, which is why her beliefs might be correlated. But then, this is exactly what happens in a CBE, viz., players are correct in their marginal beliefs about every opponent, but perhaps wrong when assessing the possibility of their opponents' strategies being correlated.

5.2. Rationalizability

In this section we study the relationship between CBE and the different forms of rationalizability. First, we relate CBE to correlated rationalizability, and subsequently to independent rationalizability.

There are two equivalent definitions of correlated rationalizability. First, according to the iterative definition a strategy profile is correlated rationalizable whenever it survives the iterated elimination of strictly dominated strategies (IESDS).[7] Alternatively, according to the fixed point definition a strategy profile is correlated rationalizable if it belongs to a product of justifiable strategy sets, with the property that each justifiable strategy of each player is a best response to a (possibly correlated) belief over the opponents' justifiable strategy profiles (Brandenburger and Dekel, 1987). Formally, a set of justifiable strategies is called best response set and is defined as follows: Consider some $D_i \subseteq A_i$ for each player $i \in I$ with the property that, for every $a_i \in D_i$ there exists a conjecture $\mu_i \in \Delta\big(\bigtimes_{j\neq i} D_j\big)$ such that $a_i \in BR_i(\mu_i)$. Then, we say that $D_1 \times \cdots \times D_n$ satisfies the best response property (with each D_i being a best response set), and every $(\sigma_1, \dots, \sigma_n) \in \bigtimes_{i\in I} \Delta(D_i)$ is called ***correlated rationalizable***.

It is well-known that every NE is a correlated rationalizable strategy profile. In fact, this is also true for every CBE, as the supports of the mixed strategies played in a CBE have the best response property, as shown below.

Proposition 4. *Every CBE is correlated rationalizable.*

Clearly, the converse is not true, e.g., in two player games it is often the case that Nash equilibria form a strict subset of the set of correlated rationalizable strategy profiles.

In general, notice that the (fixed point) definition of correlated rationalizability incorporates two assumptions, viz., each player is rational, and moreover each player correctly believes that every opponent plays a justifiable strategy. In this sense, CBE lies between NE and correlated rationalizability. Indeed, CBE postulates only CMB, whereas correlated rationalizability does not require eiether IB or CMB.

Now, we turn our attention to independent rationalizability, which strengthens correlated rationalizability in the sense that it requires each player to have independent beliefs (Bernheim, 1984; Pearce, 1984). Formally, independent rationalizability is defined as follows: Consider some $D_i \subseteq A_i$

[7] Recall that IESDS yields the strategy profiles that can be played under rationality and common belief in rationality (Brandenburger and Dekel, 1987; Tan and Werlang, 1988).

for each player $i \in I$ with the property that, for every $a_i \in D_i$ there is some $\mu_i \in \Pi\big(\times_{j \neq i} D_j\big)$ such that $a_i \in BR_i(\mu_i)$. Then, we say that $D_1 \times \cdots \times D_n$ satisfies the independent best response property (with each D_i being an independent best response set), and every $(\sigma_1, \ldots, \sigma_n) \in \times_{i \in I} \Delta(D_i)$ is said to be ***independently rationalizable***.

It is well known that there is a monotonic relationship in the strategy profiles that are predicted by NE, independent rationalizability and correlated rationalizability, in the respective order. Indeed, the following figure illustrates the relationship between the different solution concepts that we have discussed so far, together with the additional implicit condition that needs to be imposed in order to go from one to the other.

$$\begin{array}{ccc} \boxed{\text{Correlated rationalizability}} & \overset{(+\text{IB})}{\Longrightarrow} & \boxed{\text{Independent rationalizability}} \\ \Downarrow (+\text{CMB}) & & \Downarrow (+\text{CMB}) \\ \boxed{\text{Correlated-belief equilibrium}} & \overset{(+\text{IB})}{\Longrightarrow} & \boxed{\text{Nash equilibrium}} \end{array}$$

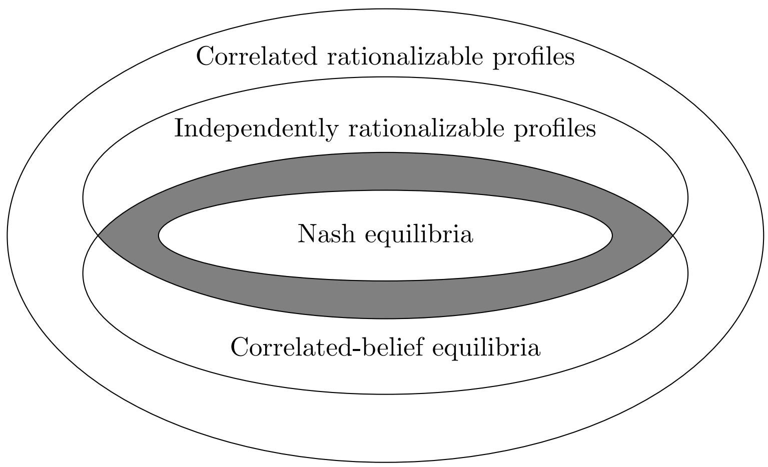
What is not straightforward from the previous figure is the relationship between independent rationalizability and CBE, as the two concepts relax different assumptions of NE, viz. in comparison with NE, CBE relaxes IB, whereas independent rationalizability relaxes CMB. In fact, it turns out that none of the two concepts refines the other. Indeed, it is straightforward that not every independently rationalizable strategy profile is always a CBE, e.g., in two-player games, the set of CBE coincides with the set of NE, and therefore there are games with independently rationalizable strategy profiles that are not CBE. Furthermore, it follows from the following example that there are games where a CBE is not independently rationalizable.

Example 4. Consider the following three-player game, with $I = \{\text{Ann } (a), \text{ Bob } (b), \text{ Carol } (c)\}$. Again, Ann chooses the matrix, Bob the row and Carol the column, i.e., $A_a = \{L, M, R\}$, $A_b = \{A, B\}$ and $A_c = \{C, D\}$. Furthermore, the payoffs are written in the respective order, i.e., first Ann, then Bob and then Carol. Now, consider the mixed strategy $(\sigma_a, \sigma_b, \sigma_c)$ where $\sigma_a = (1 \circledast L)$,

	L: C	L: D	M: C	M: D	R: C	R: D
A	1,1,1	0,0,0	2,2,2	1,2,2	0,2,2	1,2,2
B	0,0,0	1,1,1	1,2,2	0,2,2	1,2,2	2,2,2

$\sigma_b = (\frac{1}{2} \circledast A \ ; \ \frac{1}{2} \circledast B)$ and $\sigma_c = (\frac{1}{2} \circledast C \ ; \ \frac{1}{2} \circledast D)$, and observe that this is a CBE. Indeed, if we consider the conjectures $\mu_a = \left(\frac{1}{2} \circledast (A, C)\ ;\ \frac{1}{2} \circledast (B, D)\right)$, $\mu_b = \left(\frac{1}{2} \circledast (L, C)\ ;\ \frac{1}{2} \circledast (L, D)\right)$ and $\mu_c = \left(\frac{1}{2} \circledast (L, A)\ ;\ \frac{1}{2} \circledast (L, B)\right)$, both conditions of Definition 2 are satisfied. However, notice that $(\sigma_a, \sigma_b, \sigma_c)$ is not independently rationalizable, as the pure strategy L of Ann is not a best response to any product measure over $A_{-a} = \{A, B\} \times \{C, D\}$. $\triangleleft$

It follows from the previous discussion that the strategy profiles that can be played under the different solution concepts satisfy the inclusion relationships shown in the following figure. In general, whether these inclusions are weak or strict depends on the game in hand.



Thus, a natural question that arises at this point is whether the shaded area is empty or not. In other words, is it the case that a CBE is a NE if and only if it is independently rationalizable? It turns out that this is not the case in general. To see this, consider Example 1, and observe that every strategy profile is independently rationalizable, while at the same time there exists a CBE which is not a NE.

At first glance, this last observation looks somewhat surprising. The reason is that the conditions imposed in a CBE, together with those imposed by independent rationalizability, seem to suffice for a NE. However, if we take a closer look, we realize that there may exist some independently rationalizable CBE $(\sigma_1, \dots, \sigma_n)$ such that for some $i \in I$ the only product measure ν_i over the opponents' justifiable strategy profiles, which also satisfies $\sigma_i \in BR_i(\nu_i)$, induces different marginal conjectures than μ_i, and therefore $(\sigma_1, \dots, \sigma_n)$ is not a NE. For instance, in Example 1, the only product measures over $\{A, B\} \times \{C, D\}$ that make L a rational strategy, put probability 1 to (A, C) or probability 1 to (B, D). However, none of these product measures induces the same marginal conjectures as μ_a, and therefore the independently rationalizable CBE is not a NE.

5.3. Correlated equilibrium

So far, correlation has entered the picture only in the form of players having correlated conjectures. However, it is sometimes the case that the players' actual strategies are indeed correlated. Obviously, in this case the set of possible objective distributions (over A) increases. Having recognized this possibility Aumann introduced the concept of correlated equilibrium, which generalizes NE both in terms of strategy profiles, as well as in terms of expected payoff vectors (Aumann, 1974, 1987).

In this case, correlation takes place via some correlating random device $\big(\Omega, (\mathcal{P}_i)_{i\in I}, (\pi_i)_{i\in I}\big)$, where Ω is a finite state space and $\mathcal{P}_i$ is i's information partition over Ω.[8] The probability measure $\pi_i \in \Delta(\Omega)$ describes player i's prior beliefs. If there is some $\pi \in \Delta(\Omega)$ such that $\pi_i = \pi$ for all $i \in I$, we say that there is a common prior, and the correlating device is called objective. Otherwise, we say that the correlating device is subjective. The $\mathcal{P}_i$-measurable function $\hat{a}_i : \Omega \to A_i$ determines the pure strategy that player i undertakes upon observing the event $P_i(\omega)$, with $\hat{A}_i$ being the set of all ($\mathcal{P}_i$-measurable) contingent strategy plans. As usual, define $\hat{A} := \times_{i\in I} \hat{A}_i$ and $\hat{A}_{-i} := \times_{j\neq i} \hat{A}_j$ with typical elements $\hat{a}$ and $\hat{a}_{-i}$ respectively. Obviously, each $\hat{a} \in \hat{A}$ can be thought as a $(\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_n)$-measurable function, mapping each ω to the pure strategy profile $\big(\hat{a}_1(\omega), \ldots, \hat{a}_n(\omega)\big)$.[9]

A tuple $\big(\Omega, (\mathcal{P}_i)_{i\in I}, (\pi_i)_{i\in I}, (\hat{a}_i)_{i\in I}\big)$, consisting of a subjective (resp. objective) correlating device and a contingent strategy profile, is called a subjective (resp. objective) correlated strategy profile. For each player's prior, a correlated strategy profile induces the probability distribution $p_i := (\pi_i \circ \hat{a}^{-1}) \in \Delta(A)$ over the set of strategy profiles, i.e., player i's prior attaches probability

$$p_i(a) := \pi_i\big(\{\omega \in \Omega : \hat{a}(\omega) = a\}\big) \tag{16}$$

to each $a \in A$. Of course, p_i is not necessarily a product measure over A. In this respect it becomes obvious that the set of correlated strategy profiles induces strictly more distributions over A than the set of mixed strategy profiles does.

Player i's expected utility from $\big(\Omega, (\mathcal{P}_i)_{i\in I}, (\pi_i)_{i\in I}, (\hat{a}_i)_{i\in I}\big)$ is equal to

$$\begin{aligned} U_i(\hat{a}_i, \hat{a}_{-i}) &:= \sum_{\omega\in\Omega} \pi_i(\omega) \cdot u_i\big(\hat{a}_i(\omega), \hat{a}_{-i}(\omega)\big) \\ &= \sum_{a\in A} p_i(a) \cdot u_i(a). \end{aligned} \tag{17}$$

Then, we say that $\hat{a}_i$ is a best response to $\hat{a}_{-i}$, and we write $\hat{a}_i \in BR_i(\hat{a}_{-i})$, whenever it is the case that $U_i(\hat{a}_i, \hat{a}_{-i}) \geq U_i(\hat{a}'_i, \hat{a}_{-i})$ for all $\hat{a}'_i \in \hat{A}_i$. A subjective (resp., objective) correlated strategy

[8] Note that the state space Ω conceptually differs from the state space S defined earlier in that each $\omega \in \Omega$ corresponds to a realization of a physical randomizing device, whereas each $s \in S$ corresponds to a description of each player's strategy and belief hierarchy.

[9] As usual, $(\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_n)$ denotes the coarsest common refinement of the partitions $(\mathcal{P}_i)_{i\in I}$.

profile $\left(\Omega, (\mathcal{P}_i)_{i\in I}, (\pi_i)_{i\in I}, (\hat{a}_i)_{i\in I}\right)$ is a ***subjective (resp., objective) correlated equilibrium*** if $\hat{a}_i \in BR_i(\hat{a}_{-i})$ for all $i \in I$.

Note that in a correlated equilibrium players are allowed to have correlated beliefs. However, the (possibly correlated) beliefs are not arbitrary. Instead they are derived from the correlating device. In this respect, player i has correct beliefs about *how* the strategies of her opponents are actually correlated, whereas in a CBE players do not in general have correct beliefs about the source of correlation that they consider. This implies that in principle there seem to exist games where CBE is a weaker concept than correlated equilibrium. However, as Proposition 5 below indicates, this is not the case, viz., the strategy profiles played in an arbitrary CBE can also be played in a subjective correlated equilibrium.

Proposition 5. *Suppose that $(\sigma_1, \ldots, \sigma_n)$ is a CBE. Then, there exists a subjective correlated equilibrium $\left(\Omega, (\mathcal{P}_i)_{i\in I}, (\pi_i)_{i\in I}, (\hat{a}_i)_{i\in I}\right)$ such that* $\text{marg}_{A_i} p_i = \sigma_i$ *for all $i \in I$.*

Of course, the converse of the previous result does not hold in general, e.g., in two-player games, every CBE is a NE, and therefore there may exist correlated equilibria with the property that the marginal distributions do not form a CBE. This is not very surprising as in a correlated equilibrium – unlike what happens in a CBE – we allow players to believe that *their own strategy* is correlated with their conjecture. Yet, in order to prove Proposition 5, we construct a correlating device that rules out this type of correlation, viz., we take the prior π_i of every player to be equal to the product measure $\sigma_i \otimes \mu_i$. Thus, intuitively the predictions of a CBE can be obtained as predictions of a special class of subjective correlated equilibria, namely those that satisfy independence between each player's own strategy σ_i and own conjecture μ_i.

An important implication of Proposition 5 is that the (subjective) correlation that is incorporated in the conjectures in a CBE can be shifted to (objective) correlation induced by a correlating device in a subjective correlated equilibrium. Of course, this could be also done indirectly, by combining the well-known result of Brandenburger and Dekel (1987) with our Proposition 4. In particular, Brandenburger and Dekel (1987) proved that correlated rationalizability is equivalent to a refinement of subjective correlated equilibrium, viz., a posteriori equilibrium. Therefore, since an arbitrary CBE is correlated rationalizable (Prop. 5), it must also be the case that there exists an a posteriori equilibrium inducing the same strategies as the CBE. Still, our constructive proof of Proposition 5 clearly illustrates the differences in the degree of correlation that may appear in a CBE and in a subjective correlated equilibrium.

Finally, a natural question that arises at this point is whether the previous result also holds for an objective correlated equilibrium. As it turns out, this is not the case, as illustrated in the following example.

Example 5. Consider the following game, with $I = \{\text{Ann } (a),\ \text{Bob } (b),\ \text{Carol } (c),\ \text{David } (d)\}$. Ann chooses the matrix horizontally, Bob the matrix vertically, Carol the row and David the column, i.e., $A_a = \{X, Y\}$, $A_b = \{L, R\}$, $A_c = \{A, B\}$ and $A_d = \{C, D\}$. Moreover, the payoffs are written in the respective order, i.e., first Ann, then Bob, then Carol and then David. Now, consider the

		L		R	
		C	D	C	D
X	A	1,2,3,4	0,2,3,4	1,2,3,4	1,2,3,4
	B	0,2,3,4	1,0,3,4	1,2,3,4	1,2,3,4
		C	D	C	D
Y	A	1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
	B	1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4

mixed strategy profile $(\sigma_a, \sigma_b, \sigma_c, \sigma_d)$ with $\sigma_a = (1 \circledast X)$, $\sigma_b = (1 \circledast L)$, $\sigma_c = (\frac{2}{3} \circledast A \ ;\ \frac{1}{3} \circledast B)$ and $\sigma_d = (\frac{2}{3} \circledast C \ ;\ \frac{1}{3} \circledast D)$. Note that this strategy profile is a CBE. Indeed, if we consider the conjectures $\mu_a = \big(\frac{2}{3} \circledast (L, A, C) \ ;\ \frac{1}{3} \circledast (L, B, D)\big)$, $\mu_b = \big(\frac{1}{3} \circledast (X, A, C) \ ;\ \frac{1}{3} \circledast (X, A, D) \ ;\ \frac{1}{3} \circledast (X, B, C)\big)$, $\mu_c = \big(\frac{2}{3} \circledast (X, L, C) \ ;\ \frac{1}{3} \circledast (X, L, D)\big)$ and $\mu_d = \big(\frac{2}{3} \circledast (X, L, A) \ ;\ \frac{1}{3} \circledast (X, L, B)\big)$, both conditions of a CBE are satisfied. In fact, notice that μ_a and μ_b are the only conjectures that satisfy the required conditions for Ann and Bob respectively. Now, suppose that there exists an objective correlated equilibrium $\big(\Omega, (\mathcal{P}_i)_{i \in I}, \pi, (\hat{a}_i)_{i \in I}\big)$ with $\operatorname{marg}_{A_i} p = \sigma_i$ for all $i \in I$. Then, it is necessarily the case that Ann plays X and Bob plays L at all states $\omega \in \operatorname{Supp}(\pi)$, and therefore p assigns probability 1 to $\{X\} \times \{L\} \times \{A, B\} \times \{C, D\}$. Moreover, both Ann and Bob are rational, in the sense that X (resp. L) is a best response to $\operatorname{marg}_{A_a} p$ (resp. to $\operatorname{marg}_{A_b} p$). However, this cannot be the case as there is no probability measure $p \in \Delta(A)$ with the property that $\operatorname{marg}_{A_a} p = \mu_a$ and $\operatorname{marg}_{A_b} p = \mu_b$ hold simultaneously. Therefore, we reach a contradiction, implying that there is no objective correlated equilibrium inducing the same marginal distributions as the CBE $(\sigma_a, \sigma_b, \sigma_c, \sigma_d)$. $\triangleleft$

The intuition behind the conclusion of the previous example becomes clear if we go back to our informal discussion on the conditions that characterize the concept of correlated equilibrium. Indeed, recall that – unlike what happens in a CBE – in a correlated equilibrium players are required to have correct beliefs about *how* the strategies of the opponents are correlated, and these beliefs are actually restricted by the correlating device. Now, in the case of subjective correlated equilibrium, this restriction is not very severe as it is canceled out by the flexibility that the different priors provide. On the other hand, in an objective correlated equilibrium this flexibility disappears, and therefore, in some games, CBE may yield predictions that objective correlated equilibrium does not for any common prior, like for instance in Example 5.

A. Proof of Section 3

Proof of Proposition 1. Let $(\sigma_1, \ldots, \sigma_n)$ be a NE, and for each $i \in I$ define $\mu_i := \sigma_{-i}$. Then, $(\sigma_1, \ldots, \sigma_n)$ is a CBE, as it is the case that $\mathrm{marg}_{A_j} \mu_i = \sigma_j$ (by construction of μ_i) and $\sigma_i \in BR_i(\mu_i)$ (by the fact that $(\sigma_1, \ldots, \sigma_n)$ is a NE). □

Proof of Corollary 1. Every NE is a CBE (by Prop. 1) and a NE always exists (Nash, 1951). Therefore, a CBE always exists. □

Proof of Proposition 2. Suppose that (σ_1, σ_2) is a CBE. Then, it follows from condition (b) in Definition 2 that $\mu_i = \sigma_j$. Furthermore, it follows from $\sigma_i \in BR_i(\mu_i)$ that $\sigma_i \in BR_i(\sigma_j)$, thus implying that (σ_1, σ_2) is a NE. □

Proof of Proposition 3. Suppose that $(a_1, \ldots, a_n)$ is a CBE, implying that for every $i \in I$ there exists some $\mu_i \in \Delta(A_{-i})$ such that $\mathrm{marg}_{A_j} \mu_i(a_j) = 1$ for all $j \neq i$. Hence, it follows that $\mu_i(a_{-i}) = \prod_{j \neq i} \mathrm{marg}_{A_j} \mu_i(a_j)$, and therefore $a_i \in BR_i(a_{-i})$, which proves that $(a_1, \ldots, a_n)$ is a NE. □

B. Proofs of Section 4

Proof of Theorem 1. Fix an arbitrary $i \in I$, and define $\sigma_i \in \Delta(A_i)$ by $\sigma_i(a_i) := q\big([a_i]\big)$ for each $a_i \in A_i$. Now, observe that for every $j \neq i$, it follows by (12) that

$$\begin{aligned} q\big([a_i]\big) &= \sum_{s \in S} \tilde{\beta}_j(s)\big([a_i]\big) \cdot q(s) \\ &= \sum_{s \in S} \mathrm{marg}_{A_i} \tilde{\mu}_j(s)(a_i) \cdot q(s) \\ &= \mathrm{marg}_{A_i} \mu_j(a_i), \end{aligned}$$

thus completing the proof of part (i). Moreover, it follows directly from $s \in B(R_1 \cap \cdots \cap R_n)$ that $\tilde{a}_j(s') \in BR_j(\mu_j)$ for every $s' \in [t_i(s)]$. Furthermore, for each $a_j \in \mathrm{Supp}(\sigma_j)$ there is some $s' \in [t_i(s)]$ with $a_j(s') = a_j$. Therefore, we conclude that $\sigma_j \in BR_j(\mu_j)$, which together with (i) implies that $(\sigma_1, \ldots, \sigma_n)$ is a CBE. □

C. Proofs of Section 5

Proof of Theorem 2. For an arbitrary pure strategy profile $a_{-i} \in A_{-i}$ there exists a unique profile of signals $(m_i^j)_{j \in I \setminus \{i\}}$ such that $\psi_i^j(a_{-i}) = m_i^j$ for all $j \in I \setminus \{i\}$. Then, by the definition of ψ_i^j it is the case that

$$(\psi_i^j)^{-1}(m_i^j | a_i) = \{a_j\} \times \Big(\bigtimes_{k \in I \setminus \{i,j\}} A_k \Big)$$

for every $a_i \in A_i$. Hence, it is the case that

$$\begin{aligned} \sigma_j(a_j) &= \sigma_{-i}\big((\psi_i^j)^{-1}(m_i^j | a_i)\big), \\ (\operatorname{marg}_{A_j} \mu_i)(a_j) &= \mu_i\big((\psi_i^j)^{-1}(m_i^j | a_i)\big), \end{aligned}$$

thus implying that $\sigma_j = \operatorname{marg}_{A_j} \mu_i$ if and only if $\sigma_{-i}\big((\psi_i^j)^{-1}(m_j | a_i)\big) = \mu_i\big((\psi_i^j)^{-1}(m_j | a_i)\big)$ for all $m_j \in M_j$. The rest of the proof follows trivially by simply applying the definitions. $\square$

Proof of Proposition 4. Let $(\sigma_1, \ldots, \sigma_n)$ be a CBE, and for each $i \in I$ define $C_i := \operatorname{Supp}(\sigma_i) \subseteq A_i$. Then, by Definition 2, there is some $\mu_i \in \Delta(A_i)$ such that $\operatorname{marg}_{A_j} \mu_i = \sigma_j$, which implies that $\mu_i \in \Delta(C_{-i})$. Moreover, again by Definition 2, it is the case that $\sigma_i \in BR_i(\mu_i)$, which completes the proof. $\square$

Proof of Proposition 5. Let $\Omega := A$ and $\mathcal{P}_i := \big\{ \{a_i\} \times A_{-i} \,\big|\, a_i \in A_i \big\}$. Moreover, let $\pi_i := \sigma_i \otimes \mu_i$. Furthermore, for each $(a_1, \ldots, a_n) \in \Omega$, define i's contingent strategy by $\hat{a}_i(a_1, \ldots, a_n) := a_i$, thus completing the construction of a subjective correlated strategy profile. Now, we are going to prove that this correlated strategy profile constitutes a subjective correlated equilibrium. In fact, notice that for every $i \in I$, it is the case that

$$\begin{aligned} U_i(\hat{a}_i, \hat{a}_{-i}) &= U_i(\sigma_i, \mu_i) \\ &\geq U_i(\sigma_i', \mu_i) \end{aligned}$$

for all $\sigma_i' \in \Sigma_i$, since $(\sigma_1, \ldots, \sigma_n)$ is a CBE. Moreover, for each $\hat{a}_i' \in \hat{A}_i$ there exists some $\sigma_i' \in \Sigma_i$ such that $U_i(\hat{a}_i', \hat{a}_{-i}) = U_i(\sigma_i', \mu_i) \leq U_i(\hat{a}_i, \hat{a}_{-i})$, thus implying that the correlated strategy profile constructed above is a subjective correlated equilibrium. Finally, notice that, by construction, for every $a \in \Omega$ it is the case that $p_i(a) = \pi_i\big(\hat{a}^{-1}(a)\big) = \pi_i(a)$. Therefore, it follows that

$$\begin{aligned} \operatorname{marg}_{A_i} p_i &= \operatorname{marg}_{A_i} \pi_i \\ &= \operatorname{marg}_{A_i}(\sigma_i \otimes \mu_i) \\ &= \sigma_i, \end{aligned}$$

thus completing the proof. $\square$

References

Aliprantis, C. & Border, K. (1994). *Infinite dimensional analysis*. Springer Verlag, Berlin.

Aumann, R.J. (1974). Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics* 1, 76–96.

——— (1987). Correlated equilibrium as an expression of Bayesian rationality. *Econometrica* 55, 1–18.

Aumann, R.J. & Brandenburger, A. (1995). Epistemic conditions for Nash equilibrium. *Econometrica* 63, 1161–1180.

Bach, C.W. & Tsakas, E. (2014). Pairwise epistemic conditions for Nash equilibrium. *Games and Economic Behavior* 85, 48–59.

Barelli, P. (2009). Consistency of beliefs and epistemic conditions for Nash and correlated equilibria. *Games and Economic Behavior* 67, 363–375.

Battigalli, P. (1987). Comportamento razionale ed equilibrio nei giochi e nelle situazioni sociali. *Universitá Bocconi Dissertation.*

Battigalli, P. & Guaitoli, D. (1997). Conjectural equilibrium and rationalizability in a game with incomplete information. *Decisions, Games and Markets: Studies in Risk and Uncertainty* 8, 97–124.

Bernheim, D. (1984). Rationalizable strategic behavior. *Econometrica* 52, 1007–1028.

Brandenburger, A. & Dekel, E. (1987). Rationalizability and correlated equilibria. *Econometrica* 55, 1391–1402.

——— (1993). Hierarchies of beliefs and common knowledge. *Journal of Economic Theory* 59, 189–198.

Brandenburger, A. & Friedenberg, A. (2008). Intrinsic correlation in games. *Journal of Economic Theory* 141, 28–67.

Esponda, I. (2013). Rationalizable conjectural equilibrium: A framework for robust predictions. *Theoretical Economics* 8, 467–501.

Feinberg, Y. (2000). Characterizing common priors in the form of posteriors. *Journal of Economic Theory* 91, 127–179.

Fudenberg, D. & Levine, D. (1993). Self-confirming equilibrium. *Econometrica* 61, 523–545.

Gilli, M. (1999). On non-Nash equilibria. *Games and Economic Behavior* 27, 184–203.

Gul, F. (1998). A comment on Aumann's Bayesian view. *Econometrica* 66, 923–927.

Hahn, F. (1977). Exercises in conjectural equilibria. *Scandinavian Journal of Economics* 79, 210–226.

——— (1978). On non-walrasian equilibria. *Review of Economic Studies* 45, 1–18.

Harsanyi, J. (1967-68). Games with incomplete information played by Bayesian players, I-III. *Management Science* 14, 159–182, 320–334, 486–502.

Kalai, E. & Lehrer, E. (1993). Subjective equilibrium in repeated games. *Econometrica* 61, 1231–1240.

Mertens, J.F. & Zamir, S. (1985). Formulation of Bayesian analysis for games with incomplete information. *International Journal of Game Theory* 14, 1–29.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics* 54, 286–295.

Pearce, D.G. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica* 52, 1029–1050.

Perea, A. (2007). A one-person doxastic characterization of Nash strategies. *Synthese* 158, 251–271.

Rubinstein, A. & Wolinsky, A. (1994). Rationalizable conjectural equilibrium: Between Nash and rationalizability. *Games and Economic Behavior* 6, 299–311.

Tan, T. & Werlang, S. (1988). The Bayesian foundations of solution concepts of games. *Journal of Economic Theory* 45, 370–391.