On consensus through communication without a commonly known protocol*

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Abstract

The present paper extends the standard model of pairwise communication among Bayesian agents to cases where the structure of the communication protocol is not commonly known. We show that, even under standard strict conditions on the structure of the protocols and the nature of the transmitted signals, a consensus may never be reached if very little asymmetric information about the protocol is introduced.

Keywords: Communication protocol, common knowledge, consensus.

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1 Introduction

Common knowledge is a very central component of game theory. The concept was formalized by Aumann (1976), who showed, in his seminal paper, that if two people have the same prior beliefs, and their posteriors for an event are common knowledge, then they necessarily agree on the same posterior beliefs. Many well-known results rely on common knowledge of some elements of the game, e.g., common knowledge of rationality leads to correlated equilibrium (Aumann, 1987), common knowledge of everybody's willingness to participate in a trade plan precludes trading (Milgrom and Stokey, 1982), common knowledge of rationality and the opponents' conjectures about everybody's strategy in a normal form game suffices for Nash equilibrium (Aumann and Brandenburger, 1995).

In most cases common knowledge was a priori taken for granted. Geanakoplos and Polemarchakis (1982) were the first ones to study how common knowledge emerges in a dynamic environment where individuals start with asymmetric information. They show that if two individuals communicate their probabilistic beliefs back and forth, they will eventually agree on a — commonly known — probability assessment. Their setting has been the stepping stone for further development of models of communication in populations with Bayesian agents. The main aim of this literature is to study the conditions for reaching a consensus in groups of people through different communication mechanisms.

A usual assumption in the literature was that communication takes place through public announcement of the signals, which is quite restrictive. Parikh and Krasucki (1990) relaxed this assumption by introducing a model of pairwise private communication: They showed that under some connectedness assumption on the structure of the communication protocol — roughly, everybody talks to everybody either directly or via others — a consensus is always reached. A number of subsequent papers studied the possibility of agreeing in environments with pairwise communication, under different assumptions about the signal functions, the protocol structure, and information structure (Krasucki, 1996; Heifetz, 1996; Koessler, 2001; Houy and Menager, 2008).

A standard implicit assumption — and very crucial for the results — is that the structure of the communication protocol is commonly known, i.e., it is common knowledge who talks to whom at each period. In this paper we relax this assumption by introducing uncertainty about conversations that took place between third parties. Consider the following example: Three individuals — Ann, Bob and Carol — privately talk about the probability they assign to some event. Communication takes place as follows: Ann talks to Bob, who talks to Carol, who talks back to Bob, who talks to Ann, and so on. If this structure is commonly known, the three individuals will eventually agree on a common assessment (Krasucki, 1996). However, it is not straightforward whether this would still be the case if Carol did not know whether Ann had already talked to Bob in the first period or not.

First, notice that in order to address the previous question we need to formally model interactive knowledge about the protocol. The existing models do not serve this purpose, as the structure of the protocol is not formally an event in the state space, and therefore we cannot formally refer to knowledge or common knowledge about it. Our contributions are the following:

- ⊠ Firstly, in Section 3, we start by enlarging the state space through adding an additional dimension describing the protocol structure. This process induces a generalized state space which incorporates the description of the protocol as part of the state, and therefore allows us to express the protocol as an event; hence, knowledge about the protocol is well-defined. We continue by showing that our generalization is a natural one as it preserves the information processing that individuals implicitly used in the standard state space. Moreover (see the example in Section 3), it helps to capture parts of the agents' reasoning process that seem hard to incorporate unless one explicitly includes the protocol structure in the formal model.

- ⊠ Secondly, Theorem 1 establishes that the existing consensus results heavily rely on the implicit assumption about the protocol being common knowledge. Namely, we show that a consensus may not be reached, even if[1] (a) agents are like-minded, (b) signals are union-consistent, and (c) it is common knowledge that the protocol is fair and satisfies information exchange[2]. Furthermore, in the example we provide, even though the protocol is not commonly known, the corresponding graph[3] is: This is quite surprising as all existing results on consensus provide sufficient conditions on the structure of the graph, rather than on the protocol itself.

- ⊠ Thirdly, Corollary 1 proves that even if we strengthen the conditions of Theorem 1, by substituting union-consistency with balanced union-consistency, as it is often done in the literature (Geanakoplos, 1989; Menager, 2008), the agents may still never reach a consensus.

- ⊠ The main technical reason for the negative results above is that union-consistent signal functions on the original state space need not translate to union-consistent signal functions on the extended state space. Therefore, in Proposition 1 we provide sufficient conditions in terms of an even stronger union-consistency property (the so-called logical sure-thing principle; see Aumann et al., 2005) which does assure that, even if the protocol is not commonly known, consensus is reached eventually.

The paper is organized as follows: Section 2 introduces the (standard) notation and terminology used throughout the paper. Section 3 generalizes the state space. The main results are presented in Section 4. Section 5 contains sufficient conditions for a consensus, and relates our results to the

[1] See Section 2 for precise definitions.

[2] A protocol satisfies information exchange whenever, i talks to j if and only if j talks to i.

[3] Every protocol induces a graph summarizing how information is transmitted: Each individual corresponds to a vertex, and there is a directed edge from i to j if i talks to j infinitely often.

existing literature. Some proofs are in the appendix.

2 Notation and preliminaries

2.1 Information and knowledge

Recall the standard model of knowledge (Aumann, 1976): We consider a finite state space Ω and a finite population $N = \{1, ..., n\}$ with typical elements i and j. Every state $\omega \in \Omega$ is a complete description of all the natural facts that occur at some instance.

Every individual $i \in N$ is endowed with an ***information partition*** P_i of Ω, with $P_i(\omega)$ being the element of the information partition that contains $\omega \in \Omega$: It is the set of states that i deems possible at ω.

We define ***knowledge*** as usual: i knows a natural event $E \subseteq \Omega$ at some state ω whenever $P_i(\omega) \subseteq E$. An event $E \subseteq \Omega$ is ***common knowledge*** at ω, whenever everybody knows it, everybody knows that everybody knows it, and so on. Aumann (1976) showed that E is commonly known at ω if and only if $(P_1 \wedge \cdots \wedge P_n)(\omega) \subseteq E$, where $P_1 \wedge \cdots \wedge P_n$ denotes the finest common coarsening of the partitions and $(P_1 \wedge \cdots \wedge P_n)(\omega)$ is its element that contains ω.

2.2 Signals and consensus

Let A, with typical element α, be a finite non-empty set of signals, which contains the values of some parameter, e.g., the subjective probability assessments assigned to some event. A ***signal (action) function*** $f_i : \Omega \to A$ determines the signal that agent i transmits at every $\omega \in \Omega$. We assume that i's signal is P_i-measurable, implying that i knows her own signal: $f_i(\omega') = f_i(\omega)$ for every $\omega' \in P_i(\omega)$. A ***consensus*** has been reached at some state ω if all individuals transmit the same signal at ω, i.e., if there is some $\alpha \in A$ such that $f_i(\omega) = \alpha$ for all $i \in N$.

Agents in the population are ***like-minded*** if there is a function $f : 2^\Omega \to A$, called the virtual signal function, such that $f_i(\omega) = f(P_i(\omega))$ for every $i \in N$ and $\omega \in \Omega$. Throughout the paper, we assume that the agents are like-minded.

The function f satisfies ***union consistency***[4] (Cave, 1983) if for all non-empty, disjoint $E_1, E_2 \subseteq \Omega$ with $f(E_1) = f(E_2)$, it holds that $f(E_1 \cup E_2) = f(E_1)$. Henceforth, for illustration purposes, we assume that f is real-valued.

[4]Bacharach (1985) used the term ***sure-thing principle*** for the same property.

2.3 Communication and updating

Suppose that every individual $i \in N$ starts with a prior information partition P_i^0 over Ω, which is transparent to everybody. The information partition of agent $i \in N$ at time $t \in \mathbb{N}$ is denoted by P_i^t. At every $t \in \mathbb{N}$, a conversation between two individuals may take place: When i talks to j at t, we say that i is the sender and j the receiver (of a P_i^t-measurable signal), and we write $s_t = i$ and $r_t = j$. In this case, j updates her information from P_j^t to P_j^{t+1}.

Updating is carried out in the standard way: Let $f_i^t(\omega) = f(P_i^t(\omega))$ denote the signal that i sends if her information partition is P_i^t and the state is ω. The receiver associates every $\alpha \in A$ with the (possibly empty) class of states, $\{\omega \in \Omega : f_i^t(\omega) = \alpha\}$, which corresponds to to the event "i has said α". This type of reasoning induces a ***collection of signal-equivalent classes***. It is straightforward verifying that this collection is a partition of Ω, known as i's ***working partition*** at time t (Parikh and Krasucki, 1990), and denoted by V_i^t, with

$$V_i^t(\omega) := \{\omega' \in \Omega : f_i^t(\omega') = f_i^t(\omega)\}. \tag{1}$$

The fact that V_i^t partitions Ω implies that there is no ambiguity about the transmitted signal. That is, from j's point of view, every state corresponds to a unique signal.

Then j updates her information in the following standard way (Parikh and Krasucki, 1990): For all $\omega \in \Omega$,

$$P_j^{t+1}(\omega) = \begin{cases} P_j^t(\omega) & \text{if } j \neq r_t, \\ P_j^t(\omega) \cap V_i^t(\omega) & \text{if } j = r_t, \text{ where } i = s_t. \end{cases} \tag{2}$$

That is, the receiver hears $f_i^t(\omega)$ at ω, thus ruling out all states which are not consistent with this signal. In other words, j's information, before having heard from i, is $P_j^t(\omega)$, and upon hearing $f_i^t(\omega) = \alpha$, j updates[5] to $P_j^{t+1}(\omega) = \{\omega' \in P_j^t(\omega) : f_i^t(\omega') = \alpha\}$, by conditioning with respect to $V_i^t(\omega)$.

The sequence of senders and receivers $\{(s_t, r_t)\}_{t=0}^{\infty}$ is called a ***protocol*** and determines who talks[6] to whom at every time. The protocol induces a graph on N: There is a directed edge from i to j, if i talks to j infinitely often, i.e., if there are infinitely many $t \in \mathbb{N}$ with $(s_t, r_t) = (i, j)$.

[5] An equivalent way of writing the refining mechanism is the following

$$P_j^{t+1} = \begin{cases} P_j^t & \text{if } j \neq r_t, \\ P_j^t \vee V_i^t & \text{if } j = r_t, \text{ where } i = s_t, \end{cases}$$

where the operator $\vee$ denotes the coarsest common refinement (join) of the two partitions (Krasucki, 1996; Heifetz, 1996).

[6] We adopt the following notational convention: $s_t = r_t$ corresponds to "nobody talking to anybody at time t".

Parikh and Krasucki (1990) called a protocol ***fair*** if the graph of directed edges is strongly connected, i.e., if there is a path of directed edges which starts from some individual, passes from all the vertexes (individuals), returning to its origin. In other words, everybody communicates with everybody directly or indirectly.

A protocol satisfies ***information exchange*** if, for all distinct $i, j \in N$ with a directed edge from i to j, there is a directed edge from j to i, i.e., i talks to j infinitely often if and only if j talks back to i infinitely often. Krasucki (1996) showed that communicating union-consistent signals through a fair protocol which satisfies information exchange leads to a consensus. The underlying idea behind this result is that two agents who talk back and forth will eventually agree with each other, and therefore a consensus is guaranteed by the fairness of the protocol. In fact, if the protocol violates information exchange, a consensus may never be reached unless we impose further structure on the nature of the signals, as illustrated by Parikh and Krasucki (1990, Ex. 2).

An implicit — but crucial — assumption underlying all results in the literature, is that the protocol is commonly known among the individuals in N. What happens otherwise is not very clear. We address this question in the following sections.

3 The generalized state space

In this section, we present a case of pairwise communication without a commonly known protocol, and illustrate why the standard framework, which does not contain a description of the protocol as part of the state, may not suffice for capturing all possible contingencies.

Before moving forward with the example, let us first stress the fact that introducing uncertainty about the protocol does not affect the main principles of how people update: When an individual receives a signal $\alpha \in A$, she updates her information by conditioning her actual information set with respect to the states where — according to her — the sender would have said α. When the protocol is commonly known, the states where, according to the receiver, α would have been said, coincide with the ones where the sender actually says α. On the other hand, when there is an information asymmetry between the sender and the receiver about the actual protocol, they may interpret the signal differently, i.e., they may associate different states to α. The following example illustrates a case where the standard model is not rich enough to capture the different interpretation of the signals, thus leading to failure to update.

Example. Let $N = \{\text{Ann } (a), \text{ Bob } (b), \text{ Carol } (c)\}$ be a population of like-minded individuals, and consider the state space $\Omega = \{\omega_1, ..., \omega_4\}$, together with a uniformly distributed prior. Let the random variable $X : \Omega \to \mathbb{R}$ be such that

$$X(\omega_1) = 3, X(\omega_2) = 1 \text{ and } X(\omega_3) = X(\omega_4) = 4,$$

and let the agents communicate the conditional expectation of X, given their information. That is, for every $E \subseteq \Omega$, let

$$f(E) := \mathbb{E}[X|E], \tag{3}$$

be the union-consistent, virtual signal function. The prior information partitions over Ω are

$$\begin{aligned} P_a^0 &= \Big\{ \{\omega_1, \omega_2\}_2 \ ; \ \{\omega_3\}_4 \ ; \ \{\omega_4\}_4 \Big\} \\ P_b^0 &= \Big\{ \{\omega_1\}_3 \ ; \ \{\omega_2, \omega_3, \omega_4\}_3 \Big\} \\ P_c^0 &= \Big\{ \{\omega_1, \omega_2, \omega_3\}_{8/3} \ ; \ \{\omega_4\}_4 \Big\} \end{aligned}$$

with the indexes denoting the signals that the individuals would transmit given every information set. Suppose that the actual communication takes place according to the Round-Robin protocol, i.e., Ann talks to Bob, who talks to Carol, who talks to Ann, and so on. However, *Carol does not know whether Ann talks to Bob at the first period or not*, and this is common knowledge. The communication structure at all other periods is also commonly known.

We focus on the way Ann interprets Carol's signal at the third period. Ann knows that Carol does not know whether Ann has talked to Bob at $t = 0$ or not. Therefore, Ann knows that, *from Carol's point of view*, Bob's (non-partitional) collection of potential information sets at $t = 1$ is

$$\left\{ \begin{array}{ll} \{\omega_1\}_3 \ ; \{\omega_2\}_1 \ ; & : \text{Ann has said "2" and Bob has conditioned wrt } \{\omega_1, \omega_2\} \text{ at } t = 0 \\ \{\omega_3, \omega_4\}_4 \ ; & : \text{Ann has said "4" and Bob has conditioned wrt } \{\omega_3, \omega_4\} \text{ at } t = 0 \\ \{\omega_1\}_3 \ ; \ \{\omega_2, \omega_3, \omega_4\}_3 \Big\} & : \text{Ann has not talked and Bob has not updated at } t = 0 \end{array} \right. \tag{4}$$

This differs from $\{ \ \{\omega_1\}_3; \{\omega_2\}_1; \{\omega_3, \omega_4\}_4 \ \}$, Bob's actual information partition at $t = 1$. By (4), Ann knows that from Carol's point of view, the set of signals that Bob could possibly send are *"1"*, *"3"* and *"4"*, and therefore Ann knows that Carol's collection of information sets, after having heard Bob, is

$$\left\{ \begin{array}{ll} \{\omega_2\}_1 \ ; & : \text{Bob has said "1" and Carol has conditioned wrt } \{\omega_2\} \text{ at } t = 1 \\ \{\omega_1, \omega_2, \omega_3\}_{8/3} \ ; \{\omega_4\}_4 \ ; & : \text{Bob has said "3" and Carol has conditioned wrt } \{\omega_1, ..., \omega_4\} \text{ at } t = 1 \\ \{\omega_3\}_4 \ ; \ \{\omega_4\}_4 \Big\} & : \text{Bob has said "4" and Carol has conditioned wrt } \{\omega_3, \omega_4\} \text{ at } t = 1 \end{array} \right.$$

Now, suppose that the actual state of the world is ω_1, where Carol says *"8/3"*. According to the standard approach — described by Equation (2) — Ann would condition with respect to $\{\omega_1, \omega_2, \omega_3\}$, and therefore would not update her information. However, if we take a closer look at Ann's knowledge about the protocol, we conclude that she can infer the actual state of the world, by reasoning as follows:

Ann infers, from Carol having said *"8/3"*, that in the previous period, $t = 1$, Bob has said *"3"* to Carol. However, according to Ann, the way Carol has interpreted Bob's signal is not precise, as Carol (see (4)) has associated *"3"* with the information sets $\{\omega_1\}$ and $\{\omega_2, \omega_3, \omega_4\}$, instead of only $\{\omega_1\}$, which belongs Bob's actual information partition. This last fact is known by Ann, who uses this information and conditions with respect to $\{\omega_1\}$, thus learning the true state. $\triangleleft$

From the previous discussion, it becomes clear that the usual approach fails to fully capture Ann's reasoning at $t = 2$. The reason is that it does not take into account the fact that the receiver (Ann) interprets *"8/3"* differently than the sender (Carol), thus failing to capture Ann's updating. In order to overcome this problem, we explicitly incorporate the structure of the protocol into the state. That way we explicitly distinguish the following two events:

E_1 : Bob says *"3"* after having heard Ann saying *"2"*,

E_2 : Bob says *"3"* after not having heard anything.

The first event, E_1, corresponds to the actual protocol, according to which Ann talks to Bob at $t = 0$, and is associated with $\{\omega_1\}$, whereas E_2 corresponds to the protocol without communication between Ann and Bob, and is associated with $\{\omega_1, ..., \omega_4\}$. Obviously, the standard model does not allow Ann, who knows the actual protocol, to disregard the states associated with E_2. On the other hand, by incorporating the protocol into the state, we allow Ann to distinguish between E_1 and E_2, and therefore condition with respect to $\{\omega_1\}$ which is associated with E_1.

Formally, let Z denote a finite set of protocols, with typical element z. Let $s_t(z)$ and $r_t(z)$ denote the sender and the receiver at time t, given the protocol z. We endow Z with a ***partition*** I_i^0 for every individual. For every $z \in Z$, let $I_i^0(z)$ denote the element of I_i^0 that contains z: It is the set of protocols that i cannot distinguish from z before the communication begins. ***Knowledge of the protocol*** is defined as usual: Individual i knows the event $G \subseteq Z$ at z whenever $I_i^0(z) \subseteq G$. ***Common knowledge of the protocol*** is defined analogously.

It is intuitively straightforward that i can always distinguish between two protocols that induce different communication structure *at the times when she participates in the communication.* Formally, let $S_i(z) := \{t : s_t(z) = i\}$ and $R_i(z) := \{t : r_t(z) = i\}$ denote the times when i acts as a sender and as a receiver respectively, given the protocol z. The following assumption is imposed throughout the paper.

Assumption. For all $i \in N$, if $z' \in I_i^0(z)$, then $\big(s_t(z'), r_t(z')\big) = \big(s_t(z), r_t(z)\big)$ for every $t \in S_i(z) \cup R_i(z)$.

That is, each individual knows *(i)* when she is spoken to and by whom and, *(ii)* when she speaks and to whom.

Let $\Theta := \Omega \times Z$ be the ***generalized state space***, each element of which fully describes all natural facts and also determines the protocol structure. Every individual inherits a ***generalized information partition*** Π_i^0 over Θ, which is derived from P_i^0 and I_i^0, as follows: For each (ω, z), let

$$\Pi_i^0(\omega, z) := P_i^0(\omega) \times I_i^0(z). \tag{5}$$

The partition Π_i^0 obviously satisfies $\text{proj}_\Omega \Pi_i^0(\omega, z) = P_i^0(\omega)$ and $\text{proj}_Z \Pi_i^0(\omega, z) = I_i^0(z)$, where proj denotes the projection of a subset of Θ on the corresponding coordinate: The private information induced by Π_i^0 is consistent with the information induced by both P_i^0 and I_i^0. Obviously, when the protocol is commonly known, the Z-dimension becomes irrelevant, and we are back to the standard setting, where Θ is degenerated to Ω, and Π_i^0 is degenerated to P_i^0.

At each time $t \in \mathbb{N}$, agent $i \in N$ has the generalized information partition Π_i^t. The ***generalized signal function*** $h_i^t : \Theta \to \mathbb{R} \cup \{\emptyset\}$ is such that at every generalized state (ω, z) the value of the signal is

$$h_i^t(\omega, z) = \begin{cases} \emptyset & \text{if } i \neq s_t(z), \\ f(\text{proj}_\Omega \Pi_i^t(\omega, z)) & \text{if } i = s_t(z). \end{cases} \tag{6}$$

That is, i does not transmit any signal at (ω, z) if she is not assigned — by the protocol z — to be the sender at t. If, on the other hand, i is assigned to speak according to z, she sends the signal that corresponds to the *natural states* that cannot be ruled out at (ω, z). Observe that generalized signals contain information only about natural facts, and not about the protocol. This is because in the standard approach — described by the heuristic treatment above — which we want to describe with our model, the protocol structure is not even an event, and therefore individuals cannot formally talk about it.

Remark 1. Notice that the generalized signal function may not be union-consistent, even if the virtual signal function is. This follows from the fact that $\text{proj}_\Omega \Pi_i^t(\omega, z)$ may be a union of two information sets with a non-empty intersection. As it turns out, the failure to transfer union-consistency from the usual to the generalized signals may have important implications, such as failure to reach an agreement through bilateral communication. We return to this issue in detail in Section 5. ◁

Let W_i^t denote i's ***generalized working partition*** of Θ at t, with $W_i^t(\omega, z)$ being the element of the partition which contains (ω, z), i.e.,

$$W_i^t(\omega, z) = \{(\omega', z') \in \Theta : h_i^t(\omega', z') = h_i^t(\omega, z)\} \tag{7}$$

contains the generalized states that yield the same signal as (ω, z) at t.

Information updating takes place in the standard way (Parikh and Krasucki, 1990); see also equation (2) above: For every $(\omega, z) \in \Theta$,

$$\Pi_j^{t+1}(\omega, z) = \begin{cases} \Pi_j^t(\omega, z) & \text{if } j \neq r_t(z), \\ \Pi_j^t(\omega, z) \cap W_i^t(\omega, z) & \text{if } j = r_t(z), \text{ where } i = s_t(z). \end{cases} \tag{8}$$

That is, j updates her information only at generalized states that assign her to be the receiver. We say that a ***consensus*** has been reached at (ω, z) at time t whenever $h_i^t(\omega, z) = \alpha$ for all $i \in N$.

Below, we illustrate that the generalized state space, which incorporates the protocol into the description of the state, captures Ann's reasoning at $t = 2$, thus solving the problem we presented above.

Example (cont.). Let z_1 be the actual Round-Robin protocol, and z_2 be the alternative protocol according to which Ann does not talk to Bob at $t = 0$. Ann and Bob can tell the difference, Carol can not: $I_a^0 = I_b^0 = \{ \{z_1\}; \{z_2\} \}$ and $I_c^0 = \{ \{z_1, z_2\} \}$. For notational simplicity, let $\theta_j^k := (\omega_j, z_k)$. Then, the partitions of the generalized state space become:

$$\begin{aligned} \Pi_a^0 &= \left\{ \{\theta_1^1, \theta_2^1\}_2 \; ; \; \{\theta_3^1\}_4 \; ; \; \{\theta_4^1\}_4 \; ; \; \{\theta_1^2, \theta_2^2\}_2 \; ; \; \{\theta_3^2\}_4 \; ; \; \{\theta_4^2\}_4 \right\} \\ \Pi_b^0 &= \left\{ \{\theta_1^1\}_3 \; ; \; \{\theta_2^1, \theta_3^1, \theta_4^1\}_3 \; ; \; \{\theta_1^2\}_3 \; ; \; \{\theta_2^2, \theta_3^2, \theta_4^2\}_3 \right\} \\ \Pi_c^0 &= \left\{ \{\theta_1^1, \theta_2^1, \theta_3^1, \theta_1^2, \theta_2^2, \theta_3^2\}_{8/3} \; ; \; \{\theta_4^1, \theta_4^2\}_4 \right\} \end{aligned}$$

At $t = 0$, Ann talks to Bob only according to z_1, and therefore Bob updates to

$$\Pi_b^1 = \left\{ \{\theta_1^1\}_3 \; ; \; \{\theta_2^1\}_1 \; ; \; \{\theta_3^1\}_4 \; ; \; \{\theta_4^1\}_4 \; ; \; \{\theta_1^2\}_3 \; ; \; \{\theta_2^2, \theta_3^2, \theta_4^2\}_3 \right\}.$$

At $t = 1$, Bob talks to Carol according to both protocols, and therefore Carol updates to

$$\Pi_c^2 = \left\{ \{\theta_1^1, \theta_1^2, \theta_2^2, \theta_3^2\}_{8/3} \; ; \; \{\theta_2^1\}_1 \; ; \; \{\theta_3^1\}_4 \; ; \; \{\theta_4^1\}_4 \; ; \; \{\theta_4^2\}_4 \right\}.$$

Likewise, at $t = 2$, Carol talks to Ann according to both protocols, and therefore Ann updates to

$$P_a^3 = \left\{ \{\theta_1^1\}_3 \; ; \; \{\theta_2^1\}_1 \; ; \; \{\theta_3^1\}_4 \; ; \; \{\theta_4^1\}_4 \; ; \; \{\theta_1^2, \theta_2^2\}_2 \; ; \; \{\theta_3^2\}_4 \; ; \; \{\theta_4^2\}_4 \right\}.$$

Notice that, when Ann hears *"8/3"*, she rules out the states (ω_2, z_1) and (ω_3, z_1). The reason is that Ann would deem ω_2 and ω_3 possible only if z_2 was the actual protocol. However, since she knows that this is not the case, she rules these states out, thus learning the true state (ω_1, z_1), which is consistent with our heuristic analysis of her reasoning in the Example above. ◁

4 Failing to agree when the protocol is not commonly known

Krasucki (1996) showed that if a population of individuals communicate their union-consistent signals through a fair protocol that satisfies information exchange, a consensus is eventually reached. The following result shows that this is no longer true if the protocol is not commonly known, even if it remains common knowledge that the protocol is fair and satisfies information exchange.

Theorem 1. *If the protocol is not common knowledge, then a consensus may never be reached, even if (a) agents are like-minded, (b) signals are union-consistent, and (c) it is common knowledge that the protocol is fair and satisfies information exchange.*

Proof. The following example proves the result. Let $N = \{a, b, c, d\}$, and $\Omega = \{\omega_1, ..., \omega_6\}$ together with a uniform prior. Let $X : \Omega \to \mathbb{R}$ be a random variable such that

$$X(\omega_1) = 2,\ X(\omega_2) = X(\omega_6) = 5,\ X(\omega_3) = X(\omega_5) = 8 \text{ and } X(\omega_4) = 4.$$

The individuals communicate conditional expected values of X, i.e., the union-consistent virtual signal function is

$$f(E) = \mathbb{E}[X|E],$$

for each $E \subseteq \Omega$. The prior information partitions over Ω are

$$\begin{aligned}
P_a^0 &= \Big\{ \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}_{16/3} \Big\} \\
P_b^0 &= \Big\{ \{\omega_1\}_2 \ ; \ \{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}_6 \Big\} \\
P_c^0 &= \Big\{ \{\omega_1, \omega_2, \omega_3, \omega_6\}_5 \ ; \ \{\omega_4, \omega_5\}_6 \Big\} \\
P_d^0 &= \Big\{ \{\omega_1, \omega_2, \omega_5, \omega_6\}_5 \ ; \ \{\omega_3, \omega_4\}_6 \Big\} .
\end{aligned}$$

There are two protocols differing only in the conversations that take place at $t = 0$:

	0	1	2	3	4	5	6	7	$\cdots$
z_1	$c \to a$	$a \to b$	$b \to a$	$a \to c$	$c \to a$	$a \to d$	$d \to a$	$a \to b$	$\cdots$
z_2	$d \to a$	$a \to b$	$b \to a$	$a \to c$	$c \to a$	$a \to d$	$d \to a$	$a \to b$	$\cdots$

Note that the conversations that take place at every $t \geq 1$ are common knowledge, whereas what happens at $t = 0$ is only known to a, c and d. Therefore — in line with the Assumption — a, c and d can distinguish the protocols, whereas b cannot: $I_i^0 = \{\ \{z_1\}; \{z_2\}\ \}$ if $i \in \{a, c, d\}$ and $I_b^0 = \{\ \{z_1, z_2\}\ \}$.

Let, for notational simplicity, $\theta_j^k := (\omega_j, z_k)$. The prior generalized information partitions, are depicted below:

$$\begin{aligned}
\Pi_a^0 &= \Big\{ \{\theta_1^1,\theta_2^1,\theta_3^1,\theta_4^1,\theta_5^1,\theta_6^1\}_{16/3} \ ; \ \{\theta_1^2,\theta_2^2,\theta_3^2,\theta_4^2,\theta_5^2,\theta_6^2\}_{16/3} \Big\} \\
\Pi_b^0 &= \Big\{ \{\theta_1^1,\theta_1^2\}_2 \ ; \ \{\theta_2^1,\theta_3^1,\theta_4^1,\theta_5^1,\theta_6^1,\theta_2^2,\theta_3^2,\theta_4^2,\theta_5^2,\theta_6^2\}_6 \Big\} \\
\Pi_c^0 &= \Big\{ \{\theta_1^1,\theta_2^1,\theta_3^1,\theta_6^1\}_5 \ ; \ \{\theta_4^1,\theta_5^1\}_6 \ ; \ \{\theta_1^2,\theta_2^2,\theta_3^2,\theta_6^2\}_5 \ ; \ \{\theta_4^2,\theta_5^2\}_6 \Big\} \\
\Pi_d^0 &= \Big\{ \{\theta_1^1,\theta_2^1,\theta_5^1,\theta_6^1\}_5 \ ; \ \{\theta_3^1,\theta_4^1\}_6 \ ; \ \{\theta_1^2,\theta_2^2,\theta_5^2,\theta_6^2\}_5 \ ; \ \{\theta_3^2,\theta_4^2\}_6 \Big\}
\end{aligned}$$

It is easy verifying that the generalized partitions at $t = 7$ have been updated to

$$\begin{aligned}
\Pi_a^7 &= \Big\{ \{\theta_1^1\}_2 \ ; \ \{\theta_2^1,\theta_5^1,\theta_6^1\}_6 \ ; \ \{\theta_3^1,\theta_4^1\}_6 \ ; \ \{\theta_1^2\}_2 \ ; \ \{\theta_2^2,\theta_3^2,\theta_6^2\}_6 \ ; \ \{\theta_4^2,\theta_5^2\}_6 \Big\} \\
\Pi_b^7 &= \Big\{ \{\theta_1^1,\theta_1^2\}_2 \ ; \ \{\theta_2^1,\theta_3^1,\theta_6^1,\theta_2^2,\theta_5^2,\theta_6^2\}_{13/2} \ ; \ \{\theta_4^1,\theta_5^1,\theta_3^2,\theta_4^2\}_{20/3} \Big\} \\
\Pi_c^7 &= \Big\{ \{\theta_1^1\}_2 \ ; \ \{\theta_2^1,\theta_3^1,\theta_6^1\}_6 \ ; \ \{\theta_4^1,\theta_5^1\}_6 \ ; \ \{\theta_1^2\}_2 \ ; \ \{\theta_2^2,\theta_3^2,\theta_6^2\}_6 \ ; \ \{\theta_4^2,\theta_5^2\}_6 \Big\} \\
\Pi_d^7 &= \Big\{ \{\theta_1^1\}_2 \ ; \ \{\theta_2^1,\theta_5^1,\theta_6^1\}_6 \ ; \ \{\theta_3^1,\theta_4^1\}_6 \ ; \ \{\theta_1^2\}_2 \ ; \ \{\theta_2^2,\theta_5^2,\theta_6^2\}_6 \ ; \ \{\theta_3^2,\theta_4^2\}_6 \Big\}
\end{aligned}$$

All the intermediate steps are presented in the Appendix. Notice that no updating takes place after $t = 7$, implying that a consensus is never reached at $\theta_2^1 = (\omega_2, z_1)$ for instance. □

Remark 2. As the two possible protocols differ in finitely many periods — in fact, they only differ in the first period — they correspond to the same graph. In other words, common knowledge of the graph induced by the protocol does not suffice for a consensus, i.e., even very little asymmetric information may lead to disagreement. This is quite interesting, as most existing results on consensus rely on conditions on the structure of the graph, rather than the protocol. ◁

Remark 3. The fact that the example in the proof of Theorem 1 satisfies our Assumption, implies that a possible failure to reach a consensus does not rely on individuals not knowing when or with who they communicate. ◁

The previous result technically relies on the failure to generalize union consistency to non-partitional information structures (on Ω) which may emerge, through communication and updating, in the absence of common knowledge of the protocol (see the Example above). Similar negative results have been proven in the literature of consensus and speculation, e.g., like-minded individuals with non-partitional information structures may agree to disagree even if their signals are commonly known (Geanakoplos, 1989). In these cases, the problem disappears as long as the information structure satisfies a standard balancedness condition (Geanakoplos, 1989): We say that a collection $\mathcal{B}$ of subsets of Ω is ***balanced*** whenever there is a collection of real numbers $\{\lambda_B\}_{B\in\mathcal{B}}$ such that

$$\sum_{B\in\mathcal{B}} \lambda_B \mathbf{1}_B(\omega) = 1, \text{ for all } \omega \in \bigcup_{B\in\mathcal{B}} B, \tag{9}$$

with $\mathbf{1}_B(\omega)$ denoting the indicator function. Then, we say that the virtual signal function f is ***balanced union consistent*** whenever for all balanced collections $\mathcal{B}$, if $f(B) = \alpha$ for all $B \in \mathcal{B}$ then $f(\cup_{B\in\mathcal{B}} B) = \alpha$.

Obviously, balanced union consistency is stronger than union consistency. The natural question that arises at this point is whether balanced union consistency suffices for a consensus when the protocol is not commonly known. In other words, does Theorem 1 still hold after having strengthened the sure-thing principle to balanced union consistency? The answer turns out to be positive, as shown below.

Corollary 1. *If the protocol is not common knowledge, then a consensus may never be reached, even if (a) agents are like-minded, (b) signals are balanced union-consistent, and (c) it is common knowledge that the protocol is fair and satisfies information exchange.*

The proof is a direct consequence of the following lemma, which proves that conditional expectations are balanced union consistent, implying that the virtual signal function in the proof of Theorem 1 satisfies this property, and therefore balanced union consistency is not strong enough to guarantee a consensus.

Lemma. *Let $Y : \Omega \to \mathbb{R}$ be a random variable on the finite probability space (Ω, π). Then, the virtual signal function f, defined as $f(E) := \mathbb{E}[Y|E]$ for all $E \subseteq \Omega$, is balanced union consistent.*

Proof. See the Appendix[7]. □

The previous results may be surprising, as information exchange is a very strong requirement, which leads to consensus (Krasucki, 1996). This is because, in general, the signals exchanged by two connected parties eventually become common knowledge between the two, and therefore an agreement follows trivially (Cave, 1983). However, in the absence of common knowledge of the protocol structure, the receiver of a signal may fail to interpret it unambiguously, which could in principle lead to disagreement.

5 Discussion

5.1 Sufficient conditions for consensus

The most crucial step towards providing sufficient conditions for a consensus without a commonly known protocol, is to identify the reasons behind the impossibility result of Theorem 1. Formally,

[7]This lemma is a generalization of Menager (2008, p. 726), who shows that conditional expectations are positively balanced union consistent. Recall that a collection $\mathcal{B}$ is positively balanced if there is a collection of positive reals, $\{\lambda_B\}_{B\in\mathcal{B}}$ that satisfies Equation (9).

lack of common knowledge of the protocol may lead to situations where union-consistency is not transferred from the usual space to the generalized one, e.g., when b's generalized information set is $\Pi_b^7(\theta_4^2) = \{\theta_3^1, \theta_4^1, \theta_4^2, \theta_5^2\}$, in the proof of Theorem 1, his signal is *"20/3"*, which differs from *"6"* which is the (common) signal that would be sent in either of the (disjoint) information sets $\{\theta_3^1, \theta_4^1\}$ and $\{\theta_4^2, \theta_5^2\}$, whose union is $\Pi_b^7(\theta_4^2)$. Hence, the consensus result of Krasucki (1996) does not apply in the generalized state space.

The failure to transfer union-consistency from Ω to Θ is attributed to the fact that the property holds only for disjoint subsets of Ω, e.g., in the proof of Theorem 1 it is violated for the non-disjoint events $\{\omega_3, \omega_4\}$ and $\{\omega_4, \omega_5\}$:

$$f(\{\omega_3, \omega_4\}) = f(\{\omega_4, \omega_5\}) \neq f(\{\omega_3, \omega_4, \omega_5\}).$$

Extending union consistency to non-disjoint events leads to a new concept, that of the ***logical sure-thing principle***, which is not only formally, but also conceptually different from union consistency (Aumann et al., 2005). We say that $f : 2^\Omega \to A$ satisfies the logical sure-thing principle whenever, $f(E_1 \cup E_2) = \alpha$ for all $E_1, E_2 \subseteq \Omega$ such that $f(E_1) = f(E_2) = \alpha$, even if E_1 and E_2 are not disjoint. By strengthening union consistency to the logical sure-thing principle, we can extend Krasucki's consensus result to cases without a commonly known protocol.

Proposition 1. *Consider a finite population of like-minded individuals, and let $\Theta := \Omega \times Z$ be a finite generalized state space. If the virtual signal function satisfies the logical sure-thing principle, and it is commonly known that the communication protocol is fair and satisfies information exchange, a consensus will be eventually reached.*

Proof. Let $h(E) := f(\text{proj}_\Omega E)$ for each $E \subseteq \Theta$, and suppose that two disjoint subsets $E_1, E_2 \subseteq \Theta$ are such that $h(E_1) = h(E_2) = \alpha$, which by definition implies $f(\text{proj}_\Omega E_1) = f(\text{proj}_\Omega E_2) = \alpha$. It follows, from the logical sure-thing principle, that $f(\text{proj}_\Omega E_1 \cup \text{proj}_\Omega E_2) = \alpha$, implying that

$$\begin{aligned} h(E_1 \cup E_2) &= f(\text{proj}_\Omega E_1 \cup E_2) \\ &= f(\text{proj}_\Omega E_1 \cup \text{proj}_\Omega E_2) = \alpha, \end{aligned}$$

which implies that h is union-consistent in Θ.

Let $(\omega, z) \in \Theta$, and suppose that i talks to j infinitely often according to z. Since Θ is finite, there is some $T \in N$ such that $\Pi_k^t = \Pi_k^T$ for all $t \geq T$ and every $k \in N$. Since, j does not update after T every time she hears i's signal, it follows that $W_i^T(\omega, z) \in \sigma(\Pi_j^T)$, where $\sigma(\cdot)$ denotes the σ-algebra generated by the partition. Moreover, it follows from the definition of the generalized signal function that $W_i^T(\omega, z) \in \sigma(\Pi_i^T)$. The last two points imply

$$W_i^T(\omega, z) \in \sigma(\Pi_j^T) \cap \sigma(\Pi_i^T) = \sigma(\Pi_j^T \wedge \Pi_i^T). \tag{10}$$

Since, z satisfies information exchange, it follows that j also talks to i infinitely often, and therefore

$$W_j^T(\omega, z) \in \sigma(\Pi_j^T \wedge \Pi_i^T). \tag{11}$$

Equations (10) and (11), together with the fact that $(\omega, z) \in \Pi_i^T(\omega, z)$, imply that both $h_i^T(\omega, z)$ and $h_j^T(\omega, z)$ are common knowledge between i and j:

$$(\Pi_j^T \wedge \Pi_i^T)(\omega, z) \subseteq W_i^T(\omega, z) \cap W_j^T(\omega, z).$$

Then, given the fact that h is union-consistent in Θ, it follows directly from Cave (1983) that $h_i^T(\omega, z) = h_j^T(\omega, z)$. Finally, since the graph of z is strongly connected, a consensus has been reached at (ω, z) by time T. □

Providing sufficient conditions — weaker than the logical sure-thing principle — for a consensus under asymmetric information about the protocol remains a question for future research.

5.2 Relationship to the existing literature

As we have already mentioned, the literature on communication and consensus almost unanimously assumes common knowledge of the protocol. The only attempts to depart from such an environment are those of Heifetz (1996) and Koessler (2001), who study a particular form of asymmetric information about the protocol. Namely, they allow the transmitted signals to fail to be delivered to the receiver with positive probability, which is obviously a special case of our setting.

In a more recent paper, Mueller-Frank (2010) models uncertainty about the structure of the protocol by explicitly incorporating the protocol description into the state of the world, similarly to our model. However, his results heavily rely on assuming that the individuals have a (common) prior over the generalized state space, thus allowing the signals to also embody information about the protocol. Putting this kind of additional structure to our model would lead to an agreement result, as the generalized signal functions would become union-consistent. However, we refrain from doing so, as our aim is to capture the learning process that takes place in cases similar to the one presented in Section 3. Moreover, not introducing a prior on the protocol space is also consistent with the interpretation that there is ambiguity about the protocol structure.

Appendix

Steps for proving Theorem 1. Below, we show how the generalized information partitions in the example of Section 4 evolve over time until $t = 7$. As we have already shown, after $t = 7$ no further updating occurs, implying that the individuals never reach a consensus. For the sake of compactness, we present for every $t = 1, ..., 7$ only the generalized partitions of individuals who update their information at that period.

At $t = 0$, individual c talks to a according to z_1, and d talks to a according to z_2. Therefore, a is the only one who updates her (generalized) partition to

$$\Pi_a^1 = \Big\{ \{\theta_1^1, \theta_2^1, \theta_3^1, \theta_6^1\}_5 \ ; \ \{\theta_4^1, \theta_5^1\}_6 \ ; \ \{\theta_1^2, \theta_2^2, \theta_5^2, \theta_6^2\}_5 \ ; \ \{\theta_3^2, \theta_4^2\}_6 \Big\} .$$

At $t = 1$, individual a talks to b according to both protocols, and b updates to

$$\Pi_b^2 = \Big\{ \{\theta_1^1, \theta_1^2\}_2 \ ; \ \{\theta_2^1, \theta_3^1, \theta_6^1, \theta_2^2, \theta_5^2, \theta_6^2\}_{13/2} \ ; \ \{\theta_4^1, \theta_5^1, \theta_3^2, \theta_4^2\}_{20/3} \Big\} .$$

At $t = 2$, individual b talks back to a according to both protocols, and a updates to

$$\Pi_a^3 = \Big\{ \{\theta_1^1\}_2 \ ; \ \{\theta_2^1, \theta_3^1, \theta_6^1\}_6 \ ; \ \{\theta_4^1, \theta_5^1\}_6 \ ; \ \{\theta_1^2\}_2 \ ; \ \{\theta_2^2, \theta_5^2, \theta_6^2\}_6 \ ; \ \{\theta_3^2, \theta_4^2\}_6 \Big\} .$$

At $t = 3$, individual a talks to c according to both protocols, and c updates to

$$\Pi_c^4 = \Big\{ \{\theta_1^1\}_2 \ ; \ \{\theta_2^1, \theta_3^1, \theta_6^1\}_6 \ ; \ \{\theta_4^1, \theta_5^1\}_6 \ ; \ \{\theta_1^2\}_2 \ ; \ \{\theta_2^2, \theta_3^2, \theta_6^2\}_6 \ ; \ \{\theta_4^2, \theta_5^2\}_6 \Big\} .$$

At $t = 4$, individual c talks back to a according to both protocols. Since c's generalized working partition is Π_a^4-measurable, a does not update, implying that $\Pi_a^5 = \Pi_a^4 = \Pi_a^3$. At $t = 5$, individual a talks to d according to both protocols, and d updates to

$$\Pi_d^6 = \Big\{ \{\theta_1^1\}_2 \ ; \ \{\theta_2^1, \theta_5^1, \theta_6^1\}_6 \ ; \ \{\theta_3^1, \theta_4^1\}_6 \ ; \ \{\theta_1^2\}_2 \ ; \ \{\theta_2^2, \theta_5^2, \theta_6^2\}_6 \ ; \ \{\theta_3^2, \theta_4^2\}_6 \Big\} .$$

Finally, at $t = 6$, individual d talks back to a according to both protocols. Since d's generalized working partition is Π_a^6-measurable, a does not update implying that $\Pi_a^7 = \Pi_a^6 = \Pi_a^5$.

As we already discussed, nobody updates her generalized information partition after $t = 7$, implying that $\Pi_i^t = \Pi^7$ for all $t > 7$, and all $i \in N$. Therefore, a consensus is never reached at (ω_2, z_1). □

Proof of Lemma. Let $\mathcal{B}$ be a balanced collection of events such that $\mathbb{E}[Y|B] = \alpha$ for all $B \in \mathcal{B}$, and define $B^* := \bigcup_{B \in \mathcal{B}} B$. It suffices to show that $\mathbb{E}[Y|B^*] = \alpha$.

Define the partition $\mathcal{D}$ of B^*, with typical element D, as follows: For every $\omega \in B^*$, let $\bigcap \big\{ B \in \mathcal{B} : \omega \in B \big\}$ be the element of $\mathcal{D}$ containing ω. Obviously, every $B \in \mathcal{B}$ is $\sigma(\mathcal{D})$-measurable. Then, $\mathbb{E}[Y|B^*]$ is rewritten as

$$\mathbb{E}[Y|B^*] = \sum_{D \subseteq B^*} \mathbb{E}[Y|D] \cdot \pi(D|B^*).$$

It follows from $\mathcal{B}$ being balanced that there is a collection of real numbers $\{\lambda_B\}_{B \in \mathcal{B}}$ such that

$$\sum_{B \in \mathcal{B}: D \subseteq B} \lambda_B = 1,$$

implying that

$$\begin{aligned} \mathbb{E}[Y|B^*] &= \sum_{D \subseteq B^*} \Big(\sum_{B \in \mathcal{B}: D \subseteq B} \lambda_B \Big) \mathbb{E}[Y|D] \cdot \pi(D|B^*) \\ &= \sum_{B \in \mathcal{B}} \sum_{D \subseteq B} \lambda_B \mathbb{E}[Y|D] \cdot \pi(D|B^*). \end{aligned}$$

Then, it follows from $D \subseteq B \subseteq B^*$ that $\pi(D|B^*) = \pi(D|B) \cdot \pi(B|B^*)$, implying that

$$\begin{aligned}
\mathbb{E}[Y|B^*] &= \sum_{B\in\mathcal{B}} \sum_{D\subseteq B} \lambda_B \mathbb{E}[Y|D] \cdot \pi(D|B) \cdot \pi(B|B^*) \\
&= \sum_{B\in\mathcal{B}} \lambda_B \cdot \pi(B|B^*) \sum_{D\subseteq B} \mathbb{E}[Y|D] \cdot \pi(D|B) \\
&= \sum_{B\in\mathcal{B}} \lambda_B \cdot \pi(B|B^*) \cdot \mathbb{E}[Y|B].
\end{aligned}$$

It follows from $\mathbb{E}[Y|B] = \alpha$ for all $B \in \mathcal{B}$ that

$$\begin{aligned}
\mathbb{E}[Y|B^*] &= \alpha \sum_{B\in\mathcal{B}} \lambda_B \cdot \pi(B|B^*) \\
&= \alpha \sum_{B\in\mathcal{B}} \lambda_B \sum_{D\subseteq B} \pi(B|D) \cdot \pi(D|B^*) \\
&= \alpha \sum_{B\in\mathcal{B}} \sum_{D\subseteq B} \lambda_B \cdot \pi(D|B^*) \\
&= \alpha \sum_{D\subseteq B^*} \Big(\sum_{B\in\mathcal{B}:D\subseteq B} \lambda_B \Big) \pi(D|B^*) \\
&= \alpha \sum_{D\subseteq B^*} \pi(D|B^*) \\
&= \alpha,
\end{aligned}$$

which completes the proof. □

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