On consensus through communication without a commonly known protocol

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Roadmap

Communication and consensus

Usual model of communication:

- Finite population of Bayesian individuals with private information.
- At every time $t = 0, 1, \dots$ an individual may transmit an information-dependent signal.
- Whoever hears the signal, updates their information.

Do people eventually agree on a common signal?

Agreement results

The answer is positive in many cases:

- Two individuals (Geanakoplos and Polemarchakis, 1982).
- More than two individuals, with
  - public announcement (Cave, 1983; Bacharach, 1985), or
  - private communication (Parikh and Krasucki, 1990; Krasucki, 1996).

Our motivation/contribution

Common assumption when communication is private:

The protocol is commonly known.

Example:

- Ann talks to Bob, who talks to Carol, who talks to Ann, and so on.
- At the first period, Carol does not know what Ann says to Bob, but she does know that Ann talks to Bob.

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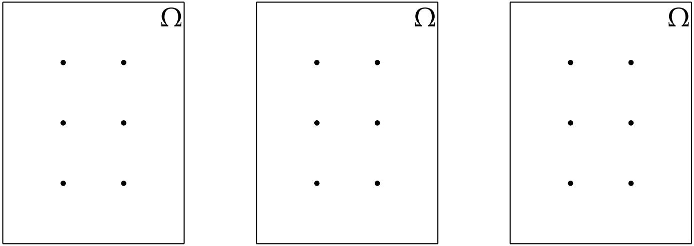
Is it crucial for the consensus result?

- Formal model of asymmetric information about the protocol.
- Without common knowledge about the protocol, they may fail to agree, even if asymmetric information is "very little".

Roadmap

Information partitions

- State space : Ω
- Finite population : $N = \{1, ..., n\}$



Information partitions

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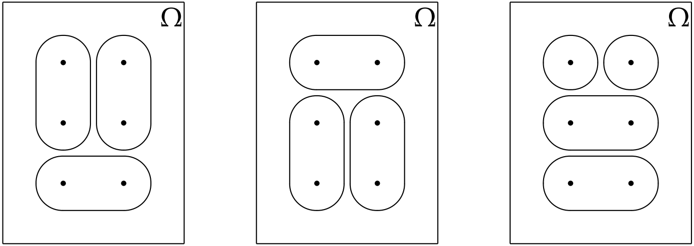
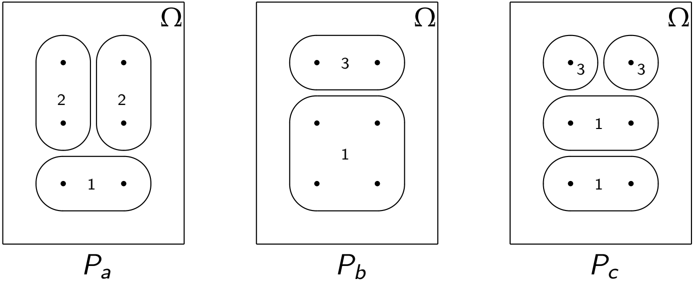


P_a P_b P_c

Signal functions

- Individual signal function : $f_i : \Omega \to \mathbb{R}$



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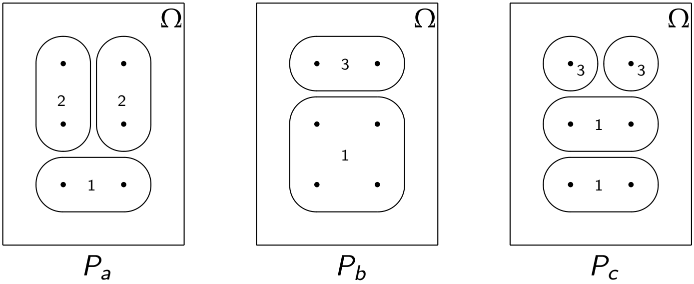

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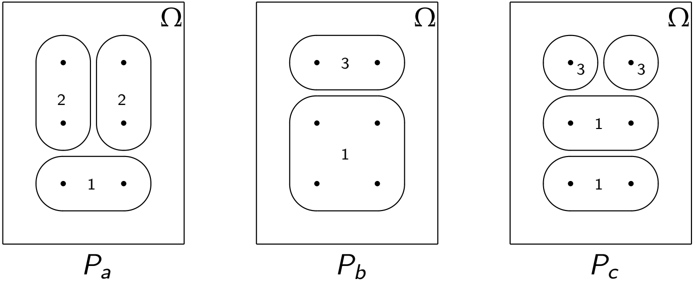

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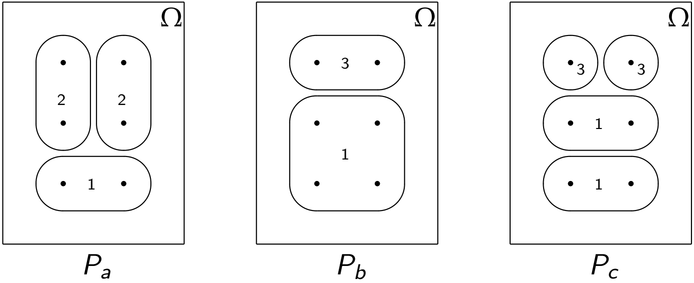

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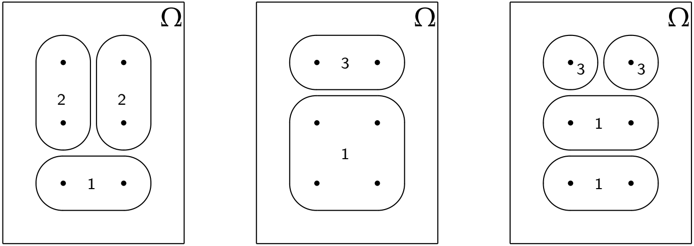
- Like-mindedness (common prior) : $f_i(\omega) = f(P_i(\omega))$
- Union-consistency : $E_1 \cap E_2 \neq \emptyset$ and $f(E_1) = f(E_2)$ imply $F(E_1 \cup E_2) = F(E_1)$



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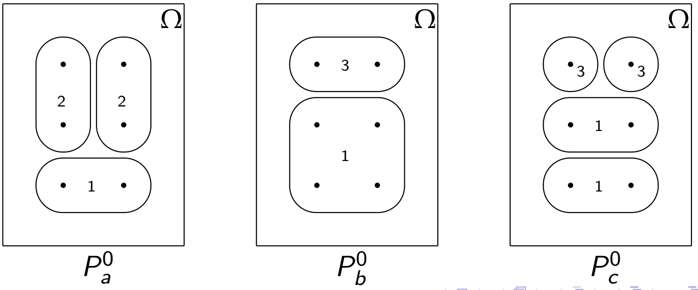
- Fair protocol : The graph is strongly connected (there is a path of directed edges which starts from some individual, passes from everybody, returning to its origin).
- Information exchange : If there is an edge from i to j, there is also an edge from j to i.

Updating

- Prior information partition : P_i^0



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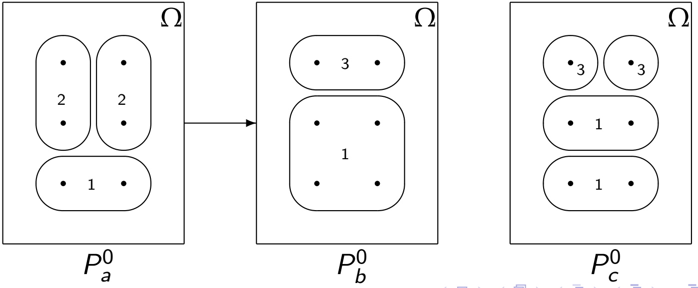

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$$P_j^{t+1}(\omega) = \begin{cases} P_j^t(\omega) & \text{if } j \neq r_t, \\ P_j^t(\omega) \cap V_i^t(\omega) & \text{if } j = r_t, \text{ where } i = s_t. \end{cases}$$

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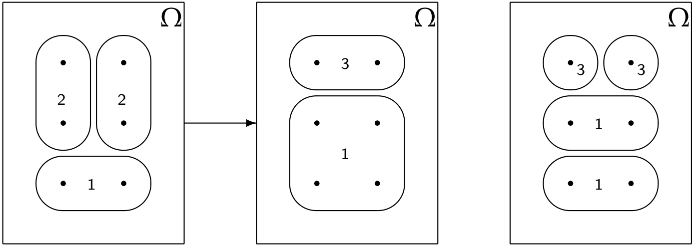

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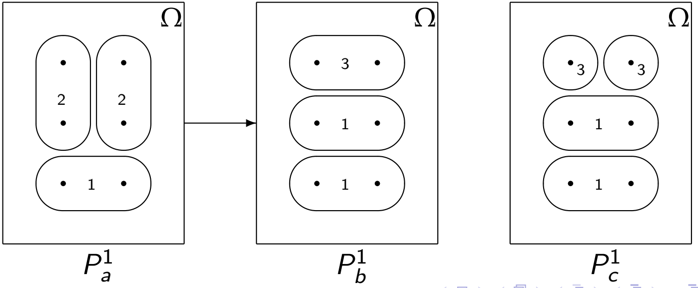
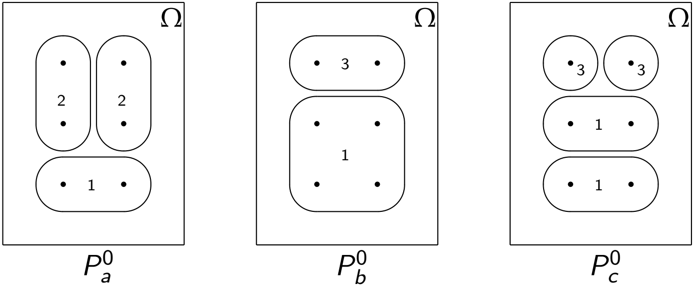


Consensus result

Proposition (Krasucki, 1996)

In a population of like-minded agents who transmit union-consistent signals through a (commonly known) fair protocol that satisfies information exchange a consensus is eventually reached.



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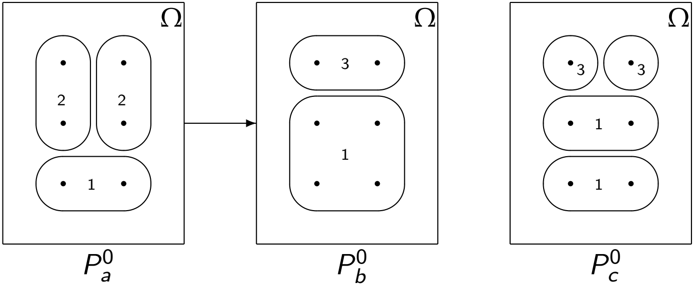

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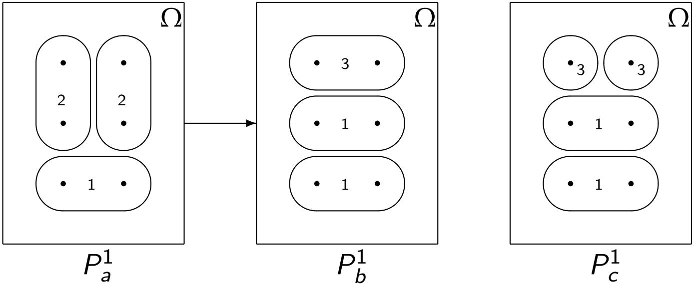

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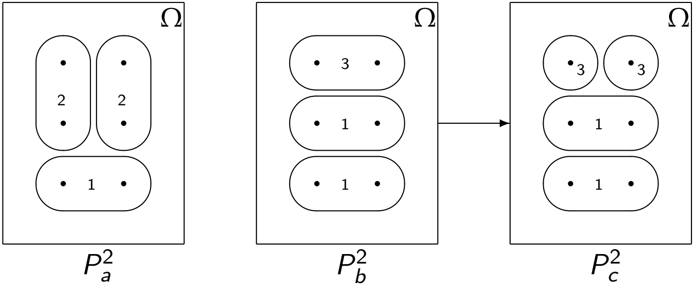

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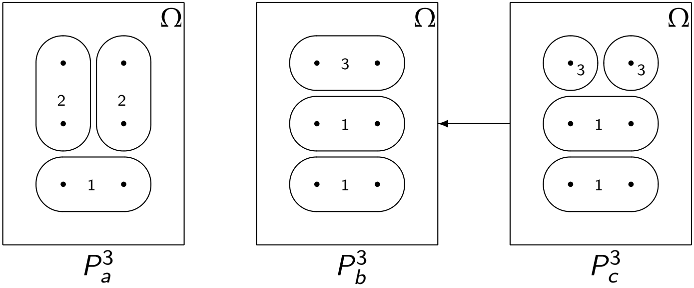

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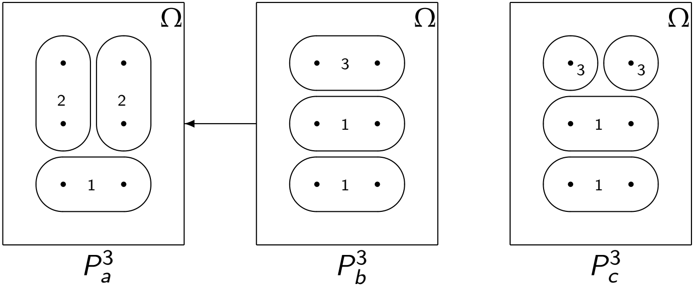

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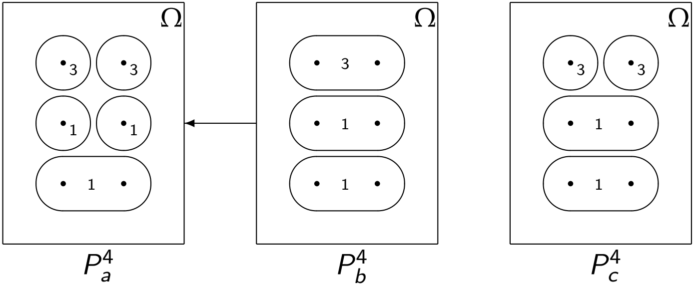

Roadmap

Without common knowledge of the protocol

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- Carol does not know whether Ann has talked to Bob at $t = 0$.

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Formal model required.

Asymmetric information about the protocol

- Finite set of protocols : Z

  Given $z \in Z$, let $\{(s_t(z), r_t(z)\}_{t=0}^{\infty}$.
- Information partition over Z : I_i^0

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  Given $z \in Z$, let $\{(s_t(z), r_t(z)\}_{t=0}^{\infty}$.

- Information partition over Z : I_i^0

  Each individual knows

  - when she is spoken to and by whom ($R_i(z) := \{t : r_t(z) = i\}$), and
  - when she speaks and to whom ($S_i(z) := \{t : s_t(z) = i\}$).

  If $z' \in I_i^0(z)$, then $(s_t(z'), r_t(z')) = (s_t(z), r_t(z))$ for every $t \in S_i(z) \cup R_i(z)$.

Generalized information partition

- Generalized state space : $\Theta = \Omega \times Z$
- Generalized prior information partition : Π_i^0

$$\boxed{\Pi_i^0(\omega, z) := \{(\omega', z') \in \Theta : \omega' \in P_i^0(\omega) \text{ and } z' \in I_i^0(z)\}}$$

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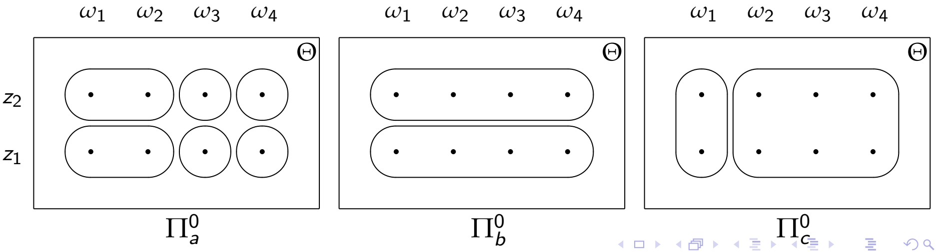

Generalized signals and updating

- Generalized signal function :

$$h_i^t(\omega, z) = \begin{cases} \emptyset & \text{if } i \neq s_t(z), \\ f(\text{proj}_\Omega \Pi_i^t(\omega, z)) & \text{if } i = s_t(z). \end{cases}$$

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where $W_i^t(\omega, z) = \{(\omega', z') \in \Theta : h_i^t(\omega', z') = h_i^t(\omega, z)\}$.

Roadmap

Main result

Theorem

If the protocol is not common knowledge, then a consensus may never be reached, even if

(a) *agents are like-minded,*
(b) *signals are union-consistent, and*
(c) *it is common knowledge that the protocol is fair and satisfies information exchange.*

Counter-example: The protocols

- Communication protocols :
  - All conversations at all $t > 0$ are common knowledge.
  - Only b does not know what happens at $t = 0$ (also common knowledge).

	0	1	2	3	4	5	6	7	$\cdots$
z_1	$c \to a$	$a \to b$	$b \to a$	$a \to c$	$c \to a$	$a \to d$	$d \to a$	$a \to b$	$\cdots$
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- Information exchange.
- Commonly known graph.

Illustration of the main result

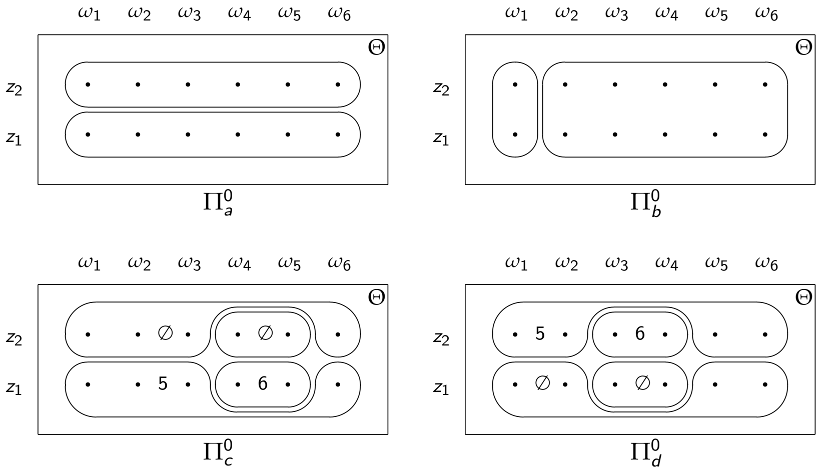

Figure: Our example

Illustration of the main result

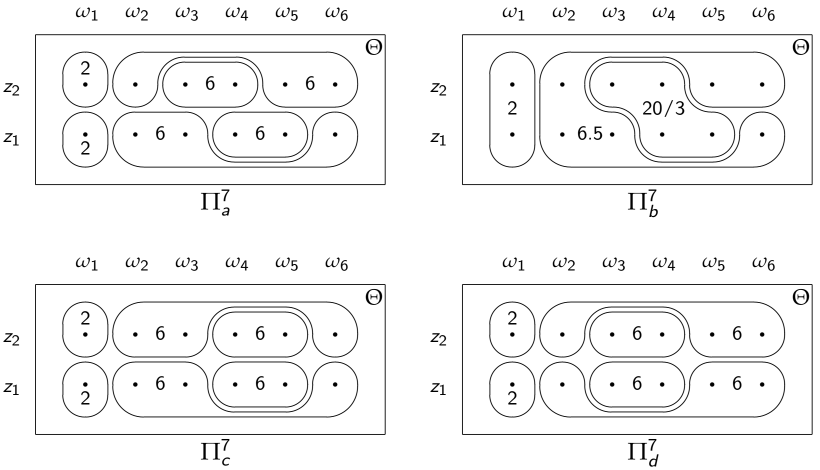

Figure: Our example

Roadmap

Tightness of the result

- Information exchange (strong requirement).
- Unique graph induced.
- Signals are not only union-consistent, but also convex.

Relationship to the existing literature

Other attempts to depart from common knowledge of the protocol:

- Heifetz (1996)
- Koessler (2001)

Special case of our model (closed eyes case), modeled by incorporating time (instead of protocols) into the state space.

Thanks for listening!!!