PAIRWISE EPISTEMIC CONDITIONS FOR NASH EQUILIBRIUM[1]

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We introduce a framework for modeling pairwise interactive beliefs and provide an epistemic foundation for Nash equilibrium in terms of pairwise epistemic conditions locally imposed on only some pairs of players. Our main result considerably weakens not only the standard sufficient conditions by Aumann and Brandenburger (1995), but also the subsequent generalization by Barelli (2009). Surprisingly, our conditions do not require nor imply mutual belief in rationality.

Keywords: Nash equilibrium, pairwise common belief, pairwise mutual belief, pairwise action-consistency, rationality, conjectures, biconnected graph, epistemic game theory.

1. INTRODUCTION

In their seminal paper, Aumann and Brandenburger (1995) provided epistemic conditions for Nash equilibrium. Accordingly, if there exists a common prior, then mutual belief in rationality and payoffs as well as common belief in each player's conjecture about the opponents' strategies imply Nash equilibrium in normal form games with more than two players. As they pointed out, in their epistemic conditions *common knowledge*[1] *enters the picture in an unexpected way*; in fact, they stressed that *what is needed is common knowledge of the players' conjectures* and *not of the players' rationality* (Aumann and Brandenburger, 1995, p. 1163). Their result challenged the widespread view that common belief in rationality is essential for Nash equilibrium. Subsequently, Polak (1999) showed that in complete information games, Aumann and Brandenburger's conditions actually do imply common belief in rationality. In a sense, his result thus restored some of the initial confidence in the importance of common belief in rationality for Nash equilibrium. More recently, Barelli (2009) generalized Aumann and Brandenburger's result by substituting the common prior assumption with the weaker action-consistency property, and common belief in conjectures with a weaker condition stating that conjectures are constant in the support of the action-consistent distribution. Thus, he provided sufficient epistemic conditions for Nash equilibrium without requiring common belief in rationality, even in complete information games.

Here, we further generalize Aumann and Brandenburger's seminal result by introducing even weaker

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[1]Aumann and Brandenburger (1995) actually use the term *knowledge* for probability-1 belief.

epistemic conditions for Nash equilibrium than those by Barelli (2009). Our results are based on introducing pairwise epistemic conditions imposed *only on some pairs of players*, contrary to the existing foundations of Aumann and Brandenburger (1995) and Barelli (2009) which correspond to pairwise epistemic conditions imposed *on all pairs of players*. Thus, our general contribution consists of providing a general framework for modeling pairwise interactive beliefs of connected agents in a graph. Such a graph can be interpreted either as an auxiliary tool used to merely weaken the customary *global* epistemic conditions to *local* ones, or as a network representing physical connections between players. In the later case, our framework opens up the possibility to connect epistemic game theory with the theory of social networks, thus enabling a link between two independently developed streams of literature. Our specific contribution consists of Theorem 1, in which we simultaneously replace (i) mutual belief in rationality with pairwise mutual belief in rationality, (ii) mutual belief in payoffs with pairwise mutual belief in payoffs, (iii) action-consistency with pairwise action-consistency, and (iv) constant conjectures in the support of the action-consistent distribution with pairwise constant conjectures in the support of the pairwise action-consistent distributions respectively, only for connected pairs of players. This difference is particularly important for large games – e.g. economies with many agents – where global epistemic conditions, such as requiring that every single player is certain that every other player is rational, can be rather demanding. In this respect, our assumptions are more plausible, as they impose pairwise conditions on relatively few pairs of players.

As a direct consequence of our main result, in Corollary 1, we also show that if a common prior exists, then pairwise mutual belief in rationality, pairwise mutual belief in payoffs and pairwise common belief in conjectures already suffice for a Nash equilibrium. The latter generalizes Aumann and Brandenburger (1995) in an orthogonal way compared to Barelli (2009). The following figure illustrates the relationship of our results to Aumann and Brandenburger (1995) and Barelli (2009).

$$\begin{array}{ccc} \boxed{\text{Theorem 1}} & \Rightarrow & \boxed{\text{Barelli (2009)}} \\ \Downarrow & & \Downarrow \\ \boxed{\text{Corollary 1}} & \Rightarrow & \boxed{\text{Aumann and Brandenburger (1995)}} \end{array}$$

Apart from introducing a new framework and from providing a more general foundation for Nash equilibrium, we also contribute to the debate about the connection between common belief in rationality and Nash equilibrium. Indeed, since our conditions are weaker than Barelli's, they do not entail common belief in rationality even in complete information games. Surprisingly however, our conditions do not even require nor imply mutual belief in rationality. Thus, we reinforce Aumann and Brandenburger's intuition about common belief in rationality *not* being essential for Nash equilibrium, by showing that even mutual belief in rationality is not a crucial component. Moreover, as our corollary indicates, the absence of common belief in rationality from the epistemic conditions for Nash equilibrium should not be necessarily linked with the lack of a common prior, but instead it could be attributed to the fact that epistemic restrictions can be local, rather than global as it was assumed in the literature so far.

2. PRELIMINARIES

2.1. *Normal form games*

Let $\big(I, (A_i)_{i\in I}, (g_i)_{i\in I}\big)$ be game in normal form, where $I = \{1, \dots, n\}$ denotes the finite set of players with typical element i, and A_i denotes the finite set of strategies, also called actions, with typical element a_i for every player $i \in I$. Moreover, define $A := \times_{i\in I} A_i$ with typical element $a := (a_1, \dots, a_n)$ and $A_{-i} := \times_{j\in I\setminus\{i\}} A_j$ with typical element $a_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. The function $g_i : A_i \times A_{-i} \to \mathbb{R}$ denotes player i's payoff function.

A probability measure $\phi_i \in \Delta(A_{-i})$ on the set of the opponents' action combinations is called a conjecture of i, with $\phi_i(a_{-i})$ signifying the probability that i attributes to the opponents playing a_{-i}. Slightly abusing notation, let $\phi_i(a_j) := \operatorname{marg}_{A_j} \phi_i(a_j)$ denote the probability that i assigns to j playing a_j. Note that it is standard to admit correlated beliefs, i.e. ϕ_i is not necessarily a product measure, hence the probability $\phi_i(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ can differ from the product $\phi_i(a_1)\cdots\phi_i(a_{i-1})\phi_i(a_{i+1})\cdots\phi_i(a_n)$ of the marginal probabilities.[2] We say that an action a_i is a best response to ϕ_i, and write $a_i \in BR_i(\phi_i)$, whenever

$$\sum_{a_{-i}\in A_{-i}} \phi_i(a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} \phi_i(a_{-i}) g_i(a'_i, a_{-i})$$

for all $a'_i \in A_i$.

A randomization over a player's actions is called mixed strategy, and is typically denoted by $\sigma_i \in \Delta(A_i)$ for all $i \in I$. Let $\Delta(A_1) \times \cdots \times \Delta(A_n)$ denote the space of mixed strategy profiles, with typical element $\sigma = (\sigma_1, \dots, \sigma_n)$. Slightly abusing terminology, we say that a pure strategy $a_i \in A_i$ is a best response to σ, and write $a_i \in BR_i(\sigma)$, whenever a_i is a best response to the product measure $\times_{j\neq i}\sigma_j$, which is an element of $\Delta(A_{-i})$. Nash's notion of equilibrium can then be defined as follows: a mixed strategy profile $(\sigma_1, \dots, \sigma_n)$ is a Nash equilibrium of the game Γ, whenever $a_i \in BR_i(\sigma)$ for all $a_i \in \operatorname{supp}(\sigma_i)$ and for all $i \in I$.

[2]Intuitively, a player's belief on his opponents' choices can be correlated, even though players choose independently from each other.

2.2. *Interactive beliefs*

Following Aumann and Brandenburger (1995), let S_i be a finite set of types for each player i, with typical element s_i.[3] As usual, let $S := S_1 \times \cdots \times S_n$ and $S_{-i} := S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$. An element $s = (s_1, \dots, s_n)$ of S is called state of the world, or simply state, while every subset of S is called an event. The event $[s_i] := \{s \in S : \mathrm{proj}_{S_i} s = s_i\}$ contains all states at which i's type is s_i. Each type $s_i \in S_i$ is associated with a probability measure over S_{-i}, called s_i's theory, which is extended to a distribution $p(\cdot; s_i) \in \Delta(S)$ over the state space, by attaching to each $E \subseteq S$ the probability that s_i's theory assigns to $\{s_{-i} \in S_{-i} : (s_i, s_{-i}) \in E\}$. In line with Aumann and Brandenburger (1995) we assume that $p([s_i]; s_i) = 1$. Therefore, the extension from s_i's theory to $p(\cdot; s_i)$ is unique.

Belief is formalized in terms of events: the set of states where agent i believes $E \subseteq S$ is defined as

$$B_i(E) := \{s \in S : p(E; s_i) = 1\}.$$

Then, it is said that i believes E at s, whenever $s \in B_i(E)$. Actually, Aumann and Brandenburger (1995) as well as subsequent papers (e.g., Polak, 1999; Barelli, 2009) use the term knowledge for probability-1 belief.

An event is mutually believed if everyone believes it. Formally, $E \subseteq S$ is *mutually believed* at s, whenever $s \in B(E)$, where

$$B(E) := \bigcap_{i \in I} B_i(E).$$

Iterating the mutual belief operator then yields higher-order mutual belief. Formally, m-th order mutual belief in E is inductively defined by $B^m(E) := B(B^{m-1}(E))$ with $B^1(E) := B(E)$. Then, an event E is commonly believed whenever everyone believes E, everyone believes that everyone believes E, etc. Formally, *common belief* in E is expressed by the event

$$CB(E) := \bigcap_{m>0} B^m(E).$$

For every player $i \in I$ an action function $\mathbf{a}_i : S \to A_i$ specifies his action at each state, and it is assumed to be S_i-measurable, i.e., $\mathbf{a}_i(s) = \mathbf{a}_i(s')$ if $\{s, s'\} \subseteq [s_i]$, implying that i attaches probability 1 to his actual strategy. The event $[a_i] := \{s \in S : \mathbf{a}_i(s) = a_i\}$ contains the states at which agent i plays a_i, while $[a] := [a_1] \cap \cdots \cap [a_n]$ and $[a_{-i}] := [a_1] \cap \cdots \cap [a_{i-1}] \cap [a_{i+1}] \cap \cdots \cap [a_n]$.

The function $\boldsymbol{\phi}_i : S \to \Delta(A_{-i})$ specifies i's *conjecture* at every state, and is defined by

$$\boldsymbol{\phi}_i(s)(a_{-i}) := p\big([a_{-i}]; s_i\big)$$

[3] Our results can be generalized to arbitrary measurable types spaces, similarly to Aumann and Brandenburger (1995, Section 6).

for each $a_{-i} \in A_{-i}$. It follows by definition that $\boldsymbol{\phi}_i$ is S_i-measurable, i.e., $\boldsymbol{\phi}_i(s) = \boldsymbol{\phi}_i(s')$ if $\{s, s'\} \subseteq [s_i]$, implying that i assign probability 1 to his actual conjecture. We define the events $[\phi_i] := \{s \in S : \boldsymbol{\phi}_i(s) = \phi_i\}$ and $[\phi_1, \ldots, \phi_n] := [\phi_1] \cap \cdots \cap [\phi_n]$.

Finally, $\mathbf{g}_i : S \times A \to \mathbb{R}$ specifies i's *payoff function* at each state of the world, and it is assumed that $\mathbf{g}_i$ is also S_i-measurable, i.e. $\mathbf{g}_i(s, a) = \mathbf{g}_i(s', a)$ if $\{s, s'\} \subseteq [s_i]$ for all $a \in A$, which implies that i attaches probability 1 to his actual payoff function. For some fixed $g_i : A \to \mathbb{R}$, let $[g_i] := \{s \in S : \mathbf{g}_i(s, a) = g_i(a),$ for all $a \in A\}$ denote the states where i's payoff function is g_i. Then, we also define $[g_1, \ldots, g_n] := [g_1] \cap \cdots \cap [g_n]$. A game is said to be of complete information if there exists $(g_1, \ldots, g_n)$ such that $[g_1, \ldots, g_n] = S$.

Furthermore, player i is *rational* at some state s, whenever he maximizes his expected payoff at s given his conjecture and payoff function. Formally,

$$R_i := \big\{ \ s \in S : \mathbf{a}_i(s) \in BR_i\big(\boldsymbol{\phi}_i(s)\big) \ \big\}$$

denotes the event that i is rational.

2.3. *Common prior and action-consistency*

A probability measure $P \in \Delta(S)$ is called a *common prior*, if for every $i \in I$ and for every $s_i \in S_i$, it i sthe case that $p(\cdot; s_i)$ coincides with the conditional distribution of P given $[s_i]$ whenever $P\big([s_i]\big) > 0$.

Recently, Barelli (2009) relaxed the common prior assumption by introducing the weaker notion of action-consistency. First, consider the set of A-measurable random variables, i.e. $\mathcal{F}_A := \{b : S \to \mathbb{R} \mid \mathbf{a}(s) = \mathbf{a}(s') \Rightarrow b(s) = b(s')\}$. Following Barelli (2009) a function in $\mathcal{F}_A$ is called action-verifiable. Then, a probability measure $\mu \in \Delta(S)$ is called *action-consistent* whenever

$$(1) \qquad \sum_{s \in S} \mu(s) b(s) = \sum_{s_i \in S_i} \mu\big([s_i]\big)\Big(\sum_{s' \in [s_i]} p(s'; s_i) b(s')\Big),$$

for every $i \in I$ and for every $b \in \mathcal{F}_A$. When A is finite, action-consistency implies

$$(2) \qquad \mu\big([a]\big) = \sum_{s_i \in S_i} p\big([a]; s_i\big)\mu\big([s_i]\big)$$

for every $i \in I$ and for every $a \in A$.

Barelli (2009) provided a characterization of action-consistency in terms of action-verifiable bets. A bet is defined as a collection $\{b_i\}_{i \in I}$ of random variables such that $\sum_{i \in I} b_i(s) = 0$ for all $s \in S$, and is called action-verifiable whenever $b_i \in \mathcal{F}_A$ for all $i \in I$. Then, Barelli (2009) showed that a probability measure is action-consistent if and only if there is no mutually beneficial action-verifiable bet among all players. Formally, $\mu \in \Delta(S)$ is action-consistent if and only if there exists no action-verifiable bet $\{b_i\}_{i \in I}$ such that $\sum_{s' \in [s_i]} p(s'; s_i) b_i(s') \geq 0$ for all $s \in \text{supp}(\mu)$ and for all $i \in I$, with at least one inequality being strict. Observe that if a common prior P exists, action-consistency is trivially satisfied by letting $\mu = P$.

If there is some action-consistent probability measure $\mu \in \Delta(S)$, then conjectures are said to be *constant in the support* of μ whenever there exists a profile of conjectures $(\phi_1, \ldots, \phi_n)$ such that $\big(\boldsymbol{\phi}_1(s), \ldots, \boldsymbol{\phi}_n(s)\big) = (\phi_1, \ldots, \phi_n)$ for all $s \in \text{supp}(\mu)$. Note that common belief of conjectures together with a common prior imply that conjectures are constant in the support of the action-consistent probability distribution.

2.4. *Epistemic foundations for Nash equilibrium*

In their seminal paper, Aumann and Brandenburger (1995) provided epistemic conditions for Nash equilibrium. Accordingly, if conjectures are derived from a common prior and are commonly believed, while at the same time rationality as well as the payoff functions are mutually believed, then all players different from i entertain the same marginal conjecture about i's action, and the marginal conjectures constitute a Nash equilibrium of the game. Formally, Aumann and Brandenburger's epistemic foundation for Nash equilibrium can be stated as follows.

Theorem A (Aumann and Brandenburger, 1995) *Let $\big(I, (A_i)_{i \in I}, (g_i)_{i \in I}\big)$ be a game and $(\phi_1, \ldots, \phi_n)$ be a tuple of conjectures. Suppose that there is a common prior that attaches positive probability to a state $s \in S$ such that $s \in B\big([g_1, \ldots, g_n]\big) \cap B(R_1 \cap \cdots \cap R_n) \cap CB\big([\phi_1, \ldots, \phi_n]\big)$. Then, there exists a mixed strategy profile $(\sigma_1, \ldots, \sigma_n)$ such that*

(i) $\text{marg}_{A_i} \phi_j = \sigma_i$ *for all* $j \in I \setminus \{i\}$,

(ii) $(\sigma_1, \ldots, \sigma_n)$ *is a Nash equilibrium of* $\big(I, (A_i)_{i \in I}, (g_i)_{i \in I}\big)$.

Subsequently, Polak (1999) showed that in complete information games, common belief in conjectures and mutual belief in rationality entail common belief in rationality. In the context of Theorem A, Polak's result implies that without common belief in rationality being present, sufficient conditions for Nash equilibrium must fail to satisfy common belief in conjectures or mutual belief in rationality or the common prior assumption.

Barelli (2009) showed that Aumann and Brandenburger's conclusions still hold even if one simultaneously substitutes the common prior assumption with action-consistency, and common belief in conjectures with constant conjectures in the support of the action-consistent distribution. Formally, Barelli's generalization of Theorem A can be stated as follows.

THEOREM B (Barelli, 2009) *Let* $\big(I,(A_i)_{i\in I},(g_i)_{i\in I}\big)$ *be a game and* $(\phi_1,\dots,\phi_n)$ *be a tuple of conjectures. Suppose that there is an action-consistent* $\mu\in\Delta(S)$ *such that* $\big(\boldsymbol{\phi}_1(s'),\dots,\boldsymbol{\phi}_n(s')\big)=(\phi_1,\dots,\phi_n)$ *for every* $s'\in\operatorname{supp}(\mu)$. *Moreover, assume that there is some state* $s\in\operatorname{supp}(\mu)$ *such that* $s\in B\big([g_1,\dots,g_n]\big)\cap B(R_1\cap\cdots\cap R_n)$. *Then, there exists a mixed strategy profile* $(\sigma_1,\dots,\sigma_n)$ *such that*

(i) $\operatorname{marg}_{A_i}\phi_j=\sigma_i$ *for all* $j\in I\setminus\{i\}$,

(ii) $(\sigma_1,\dots,\sigma_n)$ *is a Nash equilibrium of* $\big(I,(A_i)_{i\in I},(g_i)_{i\in I}\big)$.

The preceding theorem is the first epistemic foundation for Nash equilibrium without common belief in rationality in complete information games. However, note that Barelli's result still maintains Aumann and Brandenburger's assumptions of mutual belief in rationality and payoffs.

In Section 4, we weaken the conditions of Barelli (2009), and a fortiori also the ones by Aumann and Brandenburger (1995), thus obtaining a tight epistemic foundation for Nash equilibrium, without requiring neither common belief in conjectures, nor mutual belief in rationality, nor mutual belief in the payoff functions, nor action-consistency.

3. PAIRWISE EPISTEMIC CONDITIONS

3.1. *Pairwise interactive beliefs*

The standard intuitive explanation for the emergence of common belief is based on public announcement. Accordingly, once an event is publicly announced it becomes commonly believed in the sense that not only everyone believes in it, but also everyone believes that everyone in believes it, etc. Note that for mutual belief to obtain, the agents are only required to each believe in the event, and hence mere private announcements suffice.

Yet, an event may be publicly (privately) announced to some but *not all* players. For instance, an event could be publicly (privately) announced to i and j, but not to k. Pairwise common belief (pairwise mutual belief) in the event between i and j would then emerge, but not necessarily common belief (mutual belief). Due to such epistemic possibilities we now introduce pairwise interactive belief operators.

Let $E\subseteq S$ be some event and $i,j\in I$ be two players. We say that E is *pairwise mutually believed* between i and j whenever they both believe in E. Formally, pairwise mutual belief in E between i and j is denoted by the event

$$B_{i,j}(E):=B_i(E)\cap B_j(E).$$

Note that mutual belief implies pairwise mutual belief, but not conversely. We say that E is *pairwise commonly believed* between i and j whenever E is commonly believed between them. Formally, m-th order pairwise mutual belief in E is inductively defined by $B^m_{i,j}(E) := B_{i,j}(B^{m-1}_{i,j}(E))$, with $B^1_{i,j}(E) := B_{i,j}(E)$. Pairwise common belief in E between i and j is then defined as the event

$$CB_{i,j}(E) := \bigcap_{m>0} B^m_{i,j}(E).$$

Observe that common belief implies pairwise common belief, but not conversely.

3.2. *Pairwise action-consistency*

According to Barelli's characterization of action-consistency, a probability measure $\mu \in \Delta(S)$ is action-consistent if and only if there exists no mutually beneficial action-verifiable bet *among all players*. Now, suppose that, while such a bet can exist among all players, it may still be the case that there is no bilateral mutually beneficial action-verifiable bet *between i and j*. Then, although action-consistency would be violated globally, it would still hold locally between i and j. Due to such possibilities, we introduce the notion of pairwise action-consistency.

Formally, for two players $i, j \in I$, a probability measure $\mu_{i,j} \in \Delta(S)$ is called *pairwise action-consistent* between i and j if

$$\sum_{s \in S} \mu_{i,j}(s) b(s) = \sum_{s_k \in S_k} \mu_{i,j}\big([s_k]\big)\Big(\sum_{s' \in [s_k]} p(s'; s_k) b(s')\Big), \tag{3}$$

for every $k \in \{i, j\}$ and every $b \in \mathcal{F}_A$. Similar to the global case of action-consistency, pairwise action-consistency between i and j implies

$$\mu_{i,j}\big([a]\big) = \sum_{s_k \in S_k} p\big([a]; s_k\big) \mu_{i,j}\big([s_k]\big), \tag{4}$$

for every $k \in \{i, j\}$ and every $a \in A$. Observe that action-consistency trivially implies pairwise action-consistency for any pair i and j, by letting $\mu_{i,j} = \mu$.

If a probability distribution $\mu_{i,j} \in \Delta(S)$ is pairwise action-consistent between i and j, we say that *conjectures are pairwise constant in the support* of $\mu_{i,j}$ whenever there exists a tuple (ϕ_i, ϕ_j) of conjectures such that $\big(\boldsymbol{\phi}_i(s), \boldsymbol{\phi}_j(s)\big) = (\phi_i, \phi_j)$ for all $s \in \mathrm{supp}(\mu_{i,j})$.

3.3. *G-pairwise epistemic conditions*

In contrast to the standard notions of mutual belief, common belief and action-consistency, our corresponding pairwise epistemic conditions are only local, thus postulating the existence of exclusively binary relations of epistemic relevance. Formally, we represent a set of such binary relations by means of an undirected graph $G = (I, \mathcal{E})$, where the set of vertices I denotes the set of players, and edges $\mathcal{E}$ describe binary symmetric relations $(i, j) \in I \times I$ between pairs of players.

In principle, the graph G does neither enrich the epistemic model nor add any additional structure to the game whatsoever, but only provides a formal framework for imposing local pairwise restrictions on the beliefs of the players, e.g. a graph containing an edge between i and j but not between j and k can be used to model a situation where an event is pairwise mutually believed between i and j but not between j and k. Yet, there exist two complementary interpretations of G. First of all, the connectedness of two agents by an edge may be of purely epistemic character. Secondly, G may also be interpreted as a social network, in which, for instance, agents believe in the rationality of their respective neighbors only. This latter interpretation is further discussed in Section 5.

Next, some graph theoretic notions are recalled. A sequence $(i_k)_{k=1}^m$ of players is a *path* whenever $(i_k, i_{k+1}) \in \mathcal{E}$ for all $k \in \{1, \ldots, m-1\}$, i.e. in a path every two consecutive players are linked by an edge. Moreover, a graph G is called *connected* if it contains a path $(i_k)_{k=1}^m$ such that for every $i \in I$ there is some $k \in \{1, \ldots, m\}$ with $i_k = i$. Besides, G is *biconnected* if for every $i, j, k \in I$ there exists a path from i to j that does not go through k. Intuitively, a graph is biconnected if the induced subgraph that is obtained after removing an arbitrary $i \in I$ is still connected. In fact, in the context of social networks this assumptions seems quite plausible, as it states that there is no agent whose removal would disconnect the population into two components. Several well-known classes of undirected graphs are biconnected, such as for instance Hamiltonian graphs.[4] Finally, G is *complete*, if $(i, j) \in \mathcal{E}$ for all $i, j \in I$.

Specific types of pairwise epistemic conditions are now introduced.

DEFINITION 1 Let $\big(I, (A_i)_{i \in I}, (g_i)_{i \in I}\big)$ be a game, $G = (I, \mathcal{E})$ be an undirected graph, s be a state, and $(\phi_1, \ldots, \phi_n)$ be a tuple of conjectures.

- *Rationality is G-pairwise mutually believed* at s whenever $s \in B_{i,j}(R_i \cap R_j)$ for all $(i, j) \in \mathcal{E}$.
- *Payoffs are G-pairwise mutually believed* at s whenever $s \in B_{i,j}([g_i] \cap [g_j])$ for all $(i, j) \in \mathcal{E}$.
- *Conjectures are G-pairwise commonly believed* at s whenever $s \in CB_{i,j}([\phi_i] \cap [\phi_j])$ for all $(i, j) \in \mathcal{E}$.

The standard notions of mutual belief in rationality, mutual belief in payoffs and common belief in conjectures are weakened by G-pairwise mutual belief in rationality, G-pairwise mutual belief in payoffs

[4] A graph is Hamiltonian if it contains a cycle in which every vertex appears exactly once.

and G-pairwise common belief in conjectures, respectively. Formally, observe that

$$\begin{aligned} B(R_1 \cap \cdots \cap R_n) &\subseteq \bigcap_{(i,j)\in\mathcal{E}} B_{i,j}(R_i \cap R_j), \\ B\big([g_1,\ldots,g_n]\big) &\subseteq \bigcap_{(i,j)\in\mathcal{E}} B_{i,j}\big([g_i] \cap [g_j]\big), \\ CB\big([\phi_1,\ldots,\phi_n]\big) &\subseteq \bigcap_{(i,j)\in\mathcal{E}} CB_{i,j}\big([\phi_i] \cap [\phi_j]\big). \end{aligned}$$

Indeed, our concepts are weaker than the standard notions on two distinct dimensions. Firstly, the events rationality, payoffs and conjectures in Definition 1 only refer to the rationality, the payoffs and the conjectures *of the two connected player*, rather than of every $i \in I$. Secondly, our previously defined conditions impose epistemic restrictions only on the pairs of connected players in the graph, whereas standard interactive belief does so across all pairs of players.

DEFINITION 2 Let $\big(I, (A_i)_{i\in I}, (g_i)_{i\in I}\big)$ be a game, $G = (I, \mathcal{E})$ be an undirected graph, and $(\phi_1, \ldots, \phi_n)$ be a tuple of conjectures.

- *Beliefs are G-pairwise action-consistent* if there exists a collection $(\mu_{i,j})_{(i,j)\in\mathcal{E}}$ of probability measures with $\bigcap_{(i,j)\in\mathcal{E}} \mathrm{supp}(\mu_{i,j}) \neq \emptyset$ such that $\mu_{i,j}$ is pairwise action-consistent between i and j, for every $(i,j) \in \mathcal{E}$.
- If beliefs are G-pairwise action-consistent, then the *conjectures are G-pairwise constant in the supports of the action-consistent distributions* whenever $\big(\boldsymbol{\phi}_i(s), \boldsymbol{\phi}_j(s)\big) = (\phi_i, \phi_j)$ for all $s \in \mathrm{supp}(\mu_{i,j})$, and for every $(i,j) \in \mathcal{E}$.

Observe that pairwise action-consistency is not a transitive property, i.e. pairwise action-consistency between i and j and pairwise action-consistency between j and k, do not necessarily imply pairwise action-consistency between i and k. Hence, G-pairwise action-consistency does not necessarily yield action-consistency, even if G is connected.

Below, we illustrate our G-pairwise epistemic conditions by means of an example, and we also show that they are in fact weaker than the corresponding global conditions imposed by Aumann and Brandenburger (1995) and Barelli (2009).

EXAMPLE 1 Consider the game $\big(I, (A_i)_{i\in I}, (g_i)_{i\in I}\big)$, where $I = \{Alice, Bob, Claire, Donald\}$ and $A_i = \{h, \ell\}$ for all $i \in I$, and the players are abbreviated by A, B, C and D. The (commonly believed) payoff functions are as follows: *Alice* receives 1 utility unit whenever she coordinates with *Claire* and 0 otherwise, i.e. *Alice*'s payoff is equal to 1 if and only if $a_A = a_C$. *Claire* receives 1 utility unit if she chooses the same strategy as at least two other players and 0 otherwise. Finally, *Bob* and *Donald* always receive 1 utility unit, regardless of the action profile being chosen.

Now, assume the type spaces

$$\begin{aligned} S_A &= \{s_A^1(\ell), s_A^2(\ell)\}, \\ S_B &= \{s_B^1(\ell), s_B^2(h), s_B^3(\ell), s_B^4(h)\}, \\ S_C &= \{s_C^1(\ell), s_C^2(\ell)\}, \\ S_D &= \{s_D^1(\ell), s_D^2(h), s_D^3(h), s_D^4(\ell)\}, \end{aligned}$$

where the respective action in parenthesis denotes the corresponding player's action at each state given by the function $\mathbf{a}_i$. Moreover, suppose that the corresponding beliefs of each type of *Alice* are given by

$$\begin{aligned} p(\cdot; s_A^1) &= \Big(\frac{1}{4} \times (s_A^1, s_B^1, s_C^1, s_D^1) \; ; \; \frac{1}{4} \times (s_A^1, s_B^2, s_C^1, s_D^1) \; ; \; \frac{1}{4} \times (s_A^1, s_B^3, s_C^2, s_D^3) \; ; \; \frac{1}{4} \times (s_A^1, s_B^4, s_C^2, s_D^3)\Big) \\ p(\cdot; s_A^2) &= \Big(1 \times (s_A^2, s_B^1, s_C^1, s_D^2)\Big) \end{aligned}$$

with, for example, $p(\cdot; s_A^2) = \big(1 \times (s_A^2, s_B^1, s_C^1, s_D^2)\big)$ signifying that s_A^2 attaches probability 1 to the actual state being $(s_A^2, s_B^1, s_C^1, s_D^2)$. The corresponding beliefs for each type of *Bob* are given by

$$\begin{aligned} p(\cdot; s_B^1) &= \Big(\frac{1}{2} \times (s_A^1, s_B^1, s_C^1, s_D^1) \; ; \; \frac{1}{2} \times (s_A^2, s_B^1, s_C^1, s_D^2)\Big) \\ p(\cdot; s_B^2) &= \Big(\frac{1}{2} \times (s_A^1, s_B^2, s_C^1, s_D^1) \; ; \; \frac{1}{2} \times (s_A^2, s_B^2, s_C^1, s_D^2)\Big) \\ p(\cdot; s_B^3) &= \Big(\frac{1}{2} \times (s_A^1, s_B^3, s_C^2, s_D^3) \; ; \; \frac{1}{2} \times (s_A^2, s_B^3, s_C^2, s_D^4)\Big) \\ p(\cdot; s_B^4) &= \Big(\frac{1}{2} \times (s_A^1, s_B^4, s_C^2, s_D^3) \; ; \; \frac{1}{2} \times (s_A^2, s_B^4, s_C^2, s_D^4)\Big), \end{aligned}$$

the corresponding beliefs for each type of *Claire* are given by

$$\begin{aligned} p(\cdot; s_C^1) &= \Big(\frac{1}{4} \times (s_A^1, s_B^1, s_C^1, s_D^1) \; ; \; \frac{1}{4} \times (s_A^2, s_B^1, s_C^1, s_D^2) \; ; \; \frac{1}{4} \times (s_A^1, s_B^2, s_C^1, s_D^1) \; ; \; \frac{1}{4} \times (s_A^2, s_B^2, s_C^1, s_D^2)\Big) \\ p(\cdot; s_C^2) &= \Big(1 \times (s_A^1, s_B^4, s_C^2, s_D^3)\Big), \end{aligned}$$

and the corresponding beliefs for each type of *Donald* are given by

$$\begin{aligned} p(\cdot; s_D^1) &= \Big(\frac{1}{2} \times (s_A^1, s_B^1, s_C^1, s_D^1) \; ; \; \frac{1}{2} \times (s_A^1, s_B^2, s_C^1, s_D^1)\Big) \\ p(\cdot; s_D^2) &= \Big(\frac{1}{2} \times (s_A^2, s_B^1, s_C^1, s_D^2) \; ; \; \frac{1}{2} \times (s_A^2, s_B^2, s_C^1, s_D^2)\Big) \\ p(\cdot; s_D^3) &= \Big(\frac{1}{2} \times (s_A^1, s_B^3, s_C^2, s_D^3) \; ; \; \frac{1}{2} \times (s_A^1, s_B^4, s_C^2, s_D^3)\Big) \\ p(\cdot; s_D^4) &= \Big(\frac{1}{2} \times (s_A^2, s_B^3, s_C^2, s_D^4) \; ; \; \frac{1}{2} \times (s_A^2, s_B^4, s_C^2, s_D^4)\Big). \end{aligned}$$

Now, consider the Hamiltonian, and therefore biconnected graph $G = (I, \mathcal{E})$, defined by

$$\begin{aligned} I &= \{Alice, Bob, Claire, Donald\}, \\ \mathcal{E} &= \{(Alice, Bob), (Bob, Claire), (Claire, Donald), (Donald, Alice)\}. \end{aligned}$$

First, observe that the beliefs are G-pairwise action-consistent. Indeed, the probability distributions

$$\begin{aligned} \mu_{A,B} &= \Big(\frac{1}{4} \times (s_A^1, s_B^1, s_C^1, s_D^1) \; ; \; \frac{1}{4} \times (s_A^1, s_B^2, s_C^1, s_D^1) \; ; \; \frac{1}{4} \times (s_A^1, s_B^3, s_C^2, s_D^3) \; ; \; \frac{1}{4} \times (s_A^1, s_B^4, s_C^2, s_D^3)\Big) \\ &= \mu_{D,A} \end{aligned}$$

and

$$\begin{aligned}\mu_{B,C} &= \Big(\frac{1}{4}\times(s_A^1,s_B^1,s_C^1,s_D^1)\ ;\ \frac{1}{4}\times(s_A^2,s_B^1,s_C^1,s_D^2)\ ;\ \frac{1}{4}\times(s_A^1,s_B^2,s_C^1,s_D^1)\ ;\ \frac{1}{4}\times(s_A^2,s_B^2,s_C^1,s_D^2)\Big)\\ &= \mu_{C,D}\end{aligned}$$

are pairwise action-consistent between *Alice* and *Bob*, between *Bob* and *Claire*, between *Claire* and *Donald*, as well as between *Donald* and *Alice*, respectively. Moreover the intersection of their supports is non-empty as $(s_A^1,s_B^1,s_C^1,s_D^1)\in\mathrm{supp}(\mu_{i,j})$ for all $(i,j)\in\mathcal{E}$. However, by repeatedly applying Equation (2) to every $a\in A$, it obtains that if a probability measure $\mu\in\Delta(S)$ is action-consistent, then $\frac{1}{2}\mu\big([s_A^1]\big)=\mu\big([s_B^1]\big)+\mu\big([s_B^3]\big)=\mu\big([s_B^2]\big)+\mu\big([s_B^4]\big)$, which implies $\mu\big([s_A^1]\big)=1$. Hence, it follows by Equation (1) that $\sum_{s\in[s_A^1]}\big(\mu(s)-p(s;s_A^1)\big)b(s)=0$ for all $b\in\mathcal{F}_A$, thus inducing $\mu(s)=p(s;s_A^1)$ for all $s\in S$. Similarly, it can be shown that $\mu(s)=p(s;s_C^1)$ for all $s\in S$. Therefore, we obtain

$$\begin{aligned}\mu(s_A^1,s_B^3,s_C^2,s_D^3) &= p\big((s_A^1,s_B^3,s_C^2,s_D^3);s_C^1\big)\\ &\neq p\big((s_A^1,s_B^3,s_C^2,s_D^3);s_A^1\big)\\ &= \mu(s_A^1,s_B^3,s_C^2,s_D^3),\end{aligned}$$

which is a contradiction, thus implying that there exists no action-consistent probability distribution $\mu\in\Delta(S)$.

Moreover, for every $(i,j)\in\mathcal{E}$, conjectures are pairwise constant in $\mathrm{supp}(\mu_{i,j})$. For instance, at all states in $\mathrm{supp}(\mu_{A,B})=\{(s_A^1,s_B^1,s_C^1,s_D^1),(s_A^1,s_B^2,s_C^1,s_D^1),(s_A^1,s_B^3,s_C^2,s_D^3),(s_A^1,s_B^4,s_C^2,s_D^3)\}$, *Bob* assigns probability $\frac{1}{2}$ to everybody else playing ℓ and probability $\frac{1}{2}$ to *Alice* and *Claire* playing ℓ and *Donald* playing h. However, note that conjectures are not commonly believed at $(s_A^1,s_B^1,s_C^1,s_D^1)$. In fact, they are not even G-pairwise commonly believed at $(s_A^1,s_B^1,s_C^1,s_D^1)$, as at this state *Bob* attaches probability $\frac{1}{2}$ to *Alice* being certain that *Donald* will play h and probability $\frac{1}{2}$ to *Alice* being uncertain about *Donald*'s action.

Furthermore, observe that rationality is G-pairwise common belief at $(s_A^1,s_B^1,s_C^1,s_D^1)$, and therefore it is also G-pairwise mutual belief. However, rationality is not mutually believed at $(s_A^1,s_B^1,s_C^1,s_D^1)$. Indeed, *Alice* does not believe that *Claire* is rational at $(s_A^1,s_B^1,s_C^1,s_D^1)$, since ℓ is not a best response for *Claire* at $(s_A^1,s_B^4,s_C^2,s_D^3)$, because both *Bob* and *Donald* play h at this state.

Finally, note that for each $i\in I$, all players besides i share the same marginal conjecture about i's action at $(s_A^1,s_B^1,s_C^1,s_D^1)$, i.e. $\mathrm{marg}_{A_i}\,\phi_j(s_A^1,s_B^1,s_C^1,s_D^1)=\sigma_i$ for all $j\in I\setminus\{i\}$, where the mixed strategy σ_i assigns probability 1 to ℓ if $i\in\{Alice,Claire\}$ and is uniformly distributed over $\{h,\ell\}$ if $i\in\{Bob,Donald\}$. Also, $(\sigma_A,\sigma_B,\sigma_C,\sigma_D)$ constitutes a Nash equilibrium. $\triangleleft$

In the previous example, Barelli's sufficient conditions for Nash equilibrium are violated. In fact, neither action-consistency, nor mutual belief in rationality hold at $(s_A^1,s_B^1,s_C^1,s_D^1)$. Instead, G-pairwise action-consistency, together with G-pairwise constant conjectures in the supports of the action-consistent distributions, as well as G-pairwise mutual belief in rationality obtain at $(s_A^1, s_B^1, s_C^1, s_D^1)$. Moreover, observe that the conclusions of Aumann and Brandenburger (1995) as well as of Barelli (2009) do hold, i.e., all players entertain the same marginal conjectures about each of their opponents' strategy, and these marginal conjectures form a Nash equilibrium. The natural question then arises whether there exists a general relation between our G-pairwise epistemic conditions and Nash equilibrium.

4. PAIRWISE EPISTEMIC FOUNDATIONS FOR NASH EQUILIBRIUM

We now weaken the sufficient conditions for Nash equilibrium by Barelli (2009), and also by Aumann and Brandenburger (1995) by means of pairwise epistemic conditions. Indeed, the following result shows that G-pairwise mutual belief rationality, G-pairwise mutual belief of payoffs, G-pairwise action-consistency and G-pairwise constant conjectures in the support of the pairwise action-consistent probability distributions suffice for Nash equilbrium, if G is biconnected.

THEOREM 1 *Let $\big(I, (A_i)_{i \in I}, (g_i)_{i \in I}\big)$ be a game, $G = (I, \mathcal{E})$ be a biconnected graph and $(\phi_1, \ldots, \phi_n)$ be a tuple of conjectures. Suppose that for every $(i,j) \in \mathcal{E}$ there exists a pairwise action-consistent distribution $\mu_{i,j} \in \Delta(S)$ between i and j such that $\boldsymbol{\phi}_k(s') = \phi_k$ for every $k \in \{i,j\}$ and for every $s' \in \operatorname{supp}(\mu_{i,j})$. Moreover, assume that there is some state $s \in \bigcap_{(i,j)\in\mathcal{E}} \operatorname{supp}(\mu_{i,j})$ such that $s \in B_{i,j}\big([g_i] \cap [g_j]\big) \cap B_{i,j}(R_i \cap R_j)$ for all $(i,j) \in \mathcal{E}$. Then, there exists a mixed strategy profile $(\sigma_1, \ldots, \sigma_n)$ such that*

(i) $\operatorname{marg}_{A_i} \phi_j = \sigma_i$ *for all* $j \in I \setminus \{i\}$,

(ii) $(\sigma_1, \ldots, \sigma_n)$ *is a Nash equilibrium of* $\big(I, (A_i)_{i \in I}, (g_i)_{i \in I}\big)$.

The contribution of the previous result to the epistemic foundation of Nash equilibrium is twofold.

Firstly, we relax the epistemic conditions of Barelli (2009), and a fortiori also the standard conditions by Aumann and Brandenburger (1995), by no longer requiring neither mutual belief in rationality, nor mutual belief in payoffs, nor action-consistency.

Secondly, Theorem 1 offers further insight on the relation between Nash equilibrium and common belief in rationality. In fact, for many years the predominant view suggested that common belief in rationality was an essential element of Nash equilibrium. This view was challenged by Aumann and Brandenburger (1995) who required only *mutual* belief in rationality in their foundation for Nash equilibrium. Polak (1999) later observed that Aumann and Brandenburger's conditions actually do imply common belief in rationality in complete information games. In a sense, his result thus restored some of the initial confidence in the importance of common belief in rationality in the context of Nash equilibrium. More recently, Barelli (2009) provided epistemic conditions for Nash equilibrium without common belief in rationality, even in complete information games, thus confirming Aumann and Brandenburger's initial intuition about the non-necessity of common belief in rationality for Nash equilibrium. Our Theorem 1 reinforces Aumann and Brandenburger's intuition about common belief in rationality not being essential for Nash equilibrium by providing sufficient conditions for Nash equilibrium that – surprisingly – neither require nor imply mutual belief in rationality. To see this, consider Example 1, and observe that at $(s_A^1, s_B^1, s_C^1, s_D^1)$, which satisfies all the conditions of our theorem, *Alice* does not believe that *Claire* is rational, as $(s_A^1, s_B^1, s_C^1, s_D^1) \notin B_A(R_C)$.

Finally, note that Barelli's extension of Aumann and Brandenburger (1995) is based on weakening the common prior assumption. The following result, which is a direct consequence of our Theorem 1, also generalizes Aumann and Brandenburger (1995), in an orthogonal way to Barelli (2009). More specifically, it is shown that if there exists a common prior and G is a biconnected graph, then G-pairwise mutual belief in rationality, G-pairwise mutual belief in payoffs and G-pairwise common belief in conjectures suffice for Nash equilibrium.

Corollary 1 *Let $\big(I, (A_i)_{i\in I}, (g_i)_{i\in I}\big)$ be a game, $G = (I, \mathcal{E})$ be a biconnected graph and $(\phi_1, \ldots, \phi_n)$ be a tuple of conjectures. Suppose that there is a common prior attaching positive probability to some state $s \in S$ such that $s \in B_{i,j}\big([g_i] \cap [g_j]\big) \cap B_{i,j}(R_i \cap R_j) \cap CB_{i,j}\big([\phi_i] \cap [\phi_j]\big)$ for all $(i, j) \in \mathcal{E}$. Then, there exists a mixed strategy profile $(\sigma_1, \ldots, \sigma_n)$ such that*

(i) $\mathrm{marg}_{A_i} \phi_j = \sigma_i$ *for all* $j \in I \setminus \{i\}$,

(ii) $(\sigma_1, \ldots, \sigma_n)$ *is a Nash equilibrium of* $\big(I, (A_i)_{i\in I}, (g_i)_{i\in I}\big)$.

The preceding result indicates that common – or even mutual – belief in rationality may be absent from the epistemic conditions for Nash equilibrium, even if a common prior does exist.

5. DISCUSSION

5.1. *Tightness*

The assumption of the graph being biconnected is crucial for Theorem 1. Indeed, it is now shown by means of an example that the graph simply being connected does not suffice for the conclusions of Theorem 1 to obtain, even if payoffs and rationality are commonly believed, and a common prior exists. In that sense our epistemic foundations are tight.

Example 2 Consider the anti-coordination game $\big(I, (A_i)_{i\in I}, (g_i)_{i\in I}\big)$, where $I = \{Alice, Bob, Claire\}$,

$A_i = \{h, \ell\}$ for all $i \in I$, the players are abbreviated by A,B, C and D respectively, and

$$g_i(a_A, a_B, a_C) = \begin{cases} 0 & \text{if } a_A = a_B = a_C, \\ 1 & \text{otherwise.} \end{cases}$$

Now, assume the type spaces

$$\begin{aligned} S_A &= \{s_A^1(h), s_A^2(h)\}, \\ S_B &= \{s_B^1(h), s_B^2(\ell)\}, \\ S_C &= \{s_C^1(\ell)\}, \end{aligned}$$

where the respective action in parenthesis denotes the corresponding player's action at every state given by the function $\mathbf{a}_i$. Moreover, assume that the players entertain a common prior

$$P \text{ uniformly distributed over } \left\{ (s_A^1, s_B^1, s_C^1),\ (s_A^2, s_B^2, s_C^1) \right\}.$$

Let $G = (I, \mathcal{E})$ be a connected graph such that

$$\begin{aligned} I &= \{Alice, Bob, Claire\}, \\ \mathcal{E} &= \{(Alice, Bob), (Bob, Claire)\}. \end{aligned}$$

Note that at every $s \in S$, rationality is commonly believed, and conjectures are G-pairwise commonly believed. Moreover, at (s_A^1, s_B^1, s_C^1), *Alice* is certain that *Bob* chooses h and *Claire* chooses ℓ, whereas *Claire*'s conjecture attaches probability $\frac{1}{2}$ to both of her opponents playing h, and $\frac{1}{2}$ to *Alice* playing h and *Bob* playing ℓ. Therefore, *Alice* and *Claire* disagree on their marginal conjectures about *Bob*'s action, implying that the conclusion of Theorem 1 does not hold. In fact, all conditions of Theorem 1 are satisfied apart from G being biconnected. Hence, G simply being connected instead of biconnected does not suffice for Nash equilibrium. $\triangleleft$

In general, the conclusions of Theorem 1 fail for a merely connected – but not biconnected – graph, because in order for players i and j to agree on their marginal conjectures about a third player k, there must exist a path connecting i and j which does not pass through k. Otherwise, it cannot be inductively established that i and j have the same marginal conjecture about k's strategies.

5.2. *Social network interpretation*

As already mentioned, in principle the graph G does not add any additional structure to the game and is only used to describe pairwise epistemic conditions. Yet, G can be interpreted as a social network. In such a case, our conditions can be perceived as the steady state of a sequence of private communication correspondences between connected agents, e.g. similarly to Parikh and Krasucki (1990). This is particularly interesting for games with many players, e.g. large economies where agents may learn relevant personal characteristics – such as rationality, conjectures or preferences – of their neighbors *only*. However, note that our aim is not to explicitly model how pairwise interactive belief emerges in a dynamic setting, but rather to study the players' beliefs and actions, once convergence to such a state has already occurred. In any case, our graph theoretic assumption of biconnectedness appears natural in a social network context, as it admits all networks the connectivity of which does not rely on a single agent.

5.3. *Knowledge and belief*

Our sufficient conditions for Nash equilibrium are formulated in terms of probability-1 belief, instead of knowledge, similarly to most existing epistemic foundations for Nash equilibrium in the literature (Aumann and Brandenburger, 1995; Perea, 2007; Barelli, 2009). Hence, players are not required to satisfy the truth axiom, implying that players may hold false beliefs. In particular, player i may wrongly attach probability 1 to the event that j's conjecture is ϕ_j. In an earlier version of this paper, we prove Corollary 1 in a partitional model using knowledge instead of probability-1 belief (Bach and Tsakas, 2012). Note that such an approach does not restrict the state space to have a product structure as opposed to the type-based one employed here.

5.4. *Belief in an opponent's conjecture*

Already Aumann and Brandenburger (1995) recognize the conceptual difficulty in assuming belief in an opponent's conjecture. We do not intend to provide any remedy to this problematic assumption whatsoever, as we also assume that players believe in the conjectures of some of their opponents. Thus, the conceptual issue imposed by assuming belief in opponents' conjectures persists. However, we show that less belief about the opponents' conjectures is actually needed for Nash equilibrium to obtain.

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APPENDIX A: PROOFS

PROOF OF THEOREM 1: *(i).* Let $(i, j) \in \mathcal{E}$ and observe that $\boldsymbol{\phi}_i(s') = \phi_i$ for all $s' \in \operatorname{supp}(\mu_{i,j})$ by hypothesis. Then, for each $a_{-i} \in A_{-i}$, it follows that

$$\begin{aligned} p\big([a_{-i}]; s'_i\big) &= \boldsymbol{\phi}_i(s')(a_{-i}) \\ &= \phi_i(a_{-i}) \end{aligned} \tag{5}$$

for every $s' \in \operatorname{supp}(\mu_{i,j})$, and therefore

$$p\big([a_{-i}]; s'_i\big)\mu_{i,j}\big([s'_i]\big) = \phi_i(a_{-i})\mu_{i,j}\big([s'_i]\big)$$

for all $s'_i \in S_i$. Then, sum over S_i, and it follows from Equation (4) that

$$\begin{aligned} \mu_{i,j}\big([a_{-i}]\big) &= \sum_{s'_i \in S_i} p\big([a_{-i}]; s'_i\big)\mu_{i,j}\big([s'_i]\big) \\ &= \sum_{s'_i \in S_i} \phi_i(a_{-i})\mu_{i,j}\big([s'_i]\big) \\ &= \phi_i(a_{-i}) \sum_{s'_i \in S_i} \mu_{i,j}\big([s'_i]\big) \\ &= \phi_i(a_{-i}). \end{aligned} \tag{6}$$

Then, for each $k \in I \setminus \{i\}$ and every $a_k \in A_k$ it holds that

$$\mu_{i,j}\big([a_k]\big) = \phi_i(a_k). \tag{7}$$

Repeating the same steps for player j, we obtain $\mu_{i,j}\big([a_k]\big) = \phi_j(a_k)$, which implies $\operatorname{marg}_{A_k} \phi_i = \operatorname{marg}_{A_k} \phi_j$ whenever $k \in I \setminus \{i, j\}$. Finally, recall that G is a biconnected graph, implying that there is a path going through every player in $I \setminus \{k\}$. Hence, it follows from repeatedly applying the previous step that

$$\begin{aligned} \operatorname{marg}_{A_k} \phi_1 &= \operatorname{marg}_{A_k} \phi_2 = \cdots = \operatorname{marg}_{A_k} \phi_{k-1} \\ &= \operatorname{marg}_{A_k} \phi_{k+1} = \cdots = \operatorname{marg}_{A_k} \phi_n \\ &=: \sigma_k. \end{aligned}$$

Proof of (ii). Firstly, observe that for every event $E \subseteq S$,

$$p([a_i] \cap E; s_i) = p([a_i]; s_i) \cdot p(E; s_i). \tag{8}$$

Next, we show that for all $i \in I$,

$$\phi_i = \sigma_1 \times \cdots \sigma_{i-1} \times \sigma_{i+1} \times \cdots \times \sigma_n. \tag{9}$$

First, let us introduce some additional notation: For a non-empty strict subset $J \subset I$, let $A_J := \times_{k \in J} A_k$ with typical element $(a_k)_{k\in J}$. Then, for an arbitrary $(i,j)\in\mathcal{E}$ with $i\in I\setminus J$ and $j\in J$ the following equations hold:

$$\begin{aligned}
\phi_i\big((a_j)_{j\in J}\big) &= \mu_{i,j}\Big(\bigcap_{k\in J}[a_k]\Big) && \text{(by Eq.(6))}\\
&= \sum_{s_j'\in S_j} p\Big(\bigcap_{k\in J}[a_k];s_j'\Big)\mu_{i,j}\big([s_j']\big) && \text{(by Eq.(4))}\\
&= \sum_{s_j'\in S_j} p\big([a_j];s_j'\big)p\Big(\bigcap_{k\in J\setminus\{j\}}[a_k];s_j'\Big)\mu_{i,j}\big([s_j']\big) && \text{(by Eq.(8))}\\
&= \phi_j\big((a_k)_{k\in J\setminus\{j\}}\big)\sum_{s_j'\in S_j} p\big([a_j];s_j'\big)\mu_{i,j}\big([s_j']\big) && \text{(by Eq.(5))}\\
&= \phi_j\big((a_k)_{k\in J\setminus\{j\}}\big)\mu_{i,j}\big([a_j]\big) && \text{(by Eq.(4))}\\
&= \phi_j\big((a_k)_{k\in J\setminus\{j\}}\big)\phi_i(a_j) && \text{(by Eq.(7))}\\
&= \phi_\ell\big((a_k)_{k\in J\setminus\{j\}}\big)\phi_i(a_j),\ \text{ for all } \ell\in(I\setminus J)\cup\{j\} && \text{(by part } (i))
\end{aligned}\tag{10}$$

Now, consider an arbitrary $a_{-i}\in A_{-i}$. Let $J_1=I\setminus\{i\}$, and observe that since the graph is connected there exists some $j_1\in J_1$ such that $(i,j_1)\in\mathcal{E}$. Then, it follows from Eq. (10) that

$$\phi_i(a_{-i})=\phi_\ell\big((a_k)_{k\in J_1\setminus\{j_1\}}\big)\phi_i(a_{j_1})$$

for all $\ell\in\{i,j_1\}$. Now, define $J_2:=J_1\setminus\{j_1\}$, and observe that since the graph is connected there exist some $j_2\in J_2$ and $i_2\in I\setminus J_2$ such that $(i_2,j_2)\in\mathcal{E}$. Then, the previous step together with part (i) imply

$$\phi_i\big((a_k)_{k\in J_1\setminus\{j_1\}}\big)=\phi_\ell\big((a_k)_{k\in J_2\setminus\{j_2\}}\big)\phi_i(a_{j_2}).$$

for all $\ell\in\{i,j_1,j_2\}$. Continue inductively to obtain

$$\phi_i(a_{-i})=\phi_i(a_1)\cdots\phi_i(a_{i-1})\cdot\phi_i(a_{i+1})\cdots\phi_i(a_n),$$

and hence (9) obtains. Finally, $a_i\in BR_i(\phi_i)$ for all $a_i\in\mathrm{supp}(\phi_i)$ follows directly from applying Aumann and Brandenburger (1995, Lem. 4.2) to all pairs of connected players. *Q.E.D.*

Proof of Corollary 1: For an arbitrary edge $(i,j)\in\mathcal{E}$, let $F_{i,j}:=CB_{i,j}\big([\phi_i]\cap[\phi_j]\big)$. Firstly, observe that

$$\begin{aligned}
F_{i,j} &\subseteq B_i\big([\phi_i]\big) && \text{(by Aumann and Brandenburger (1995, Lem. 4.3))}\\
&= [\phi_i] && \text{(by Aumann and Brandenburger (1995, Lem. 2.6)).}
\end{aligned}\tag{11}$$

Secondly, let $\mu_{i,j}:=P(\cdot|F_{i,j})$, and note that it is a well-defined probability measure as $P(F_{i,j})>0$ holds by hypothesis. For each $k\in\{i,j\}$, it is the case that

$$\begin{aligned}
\sum_{s\in S}\mu_{i,j}(s)b(s) &= \sum_{s'\in S}P(s'|F_{i,j})b(s')\\
&= \sum_{s'\in S}\Big(\sum_{[s_k]\subseteq F_{i,j}} P\big(s'\mid[s_k]\big)P\big([s_k]\mid F_{i,j}\big)\Big)b(s') && \text{(since } F_{i,j} \text{ is } S_k\text{-measurable)}\\
&= \sum_{s'\in S}\Big(\sum_{[s_k]\subseteq F_{i,j}} p(s';s_k)\mu_{i,j}\big([s_k]\big)\Big)b(s')\\
&= \sum_{s_k\in S_k}\mu_{i,j}\big([s_k]\big)\Big(\sum_{s'\in[s_k]}p(s';s_k)b(s')\Big)
\end{aligned}$$

for every $b \in \mathcal{F}_A$. Hence, $\mu_{i,j}$ is pairwise action-consistent between i and j. Moreover, it follows from (11) that k's conjecture is constant in $\text{supp}(\mu_{i,j})$ for each $k \in \{i, j\}$. Then, the result follows directly from Theorem 1. *Q.E.D.*

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