PAIRWISE EPISTEMIC CONDITIONS FOR CORRELATED RATIONALIZABILITY[1]

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We provide a foundation for correlated rationalizability by means of pairwise epistemic conditions imposed only on some pairs of players. Indeed, we show that pairwise mutual belief, for some pairs of players, of (i) the game payoffs, (ii) rationality, and (iii) deeming possible only strategy profiles that receive positive probability by the actual conjectures suffice for correlated rationalizability when there is a common prior. Moreover, we show that our epistemic conditions do not require nor imply mutual belief of rationality. Finally, we discuss the relationship between correlated rationalizability and Nash equilibrium on the basis of the respective pairwise epistemic conditions for each of the two concepts.

KEYWORDS: Correlated rationalizability, pairwise mutual belief, rationality, conjectures, epistemic game theory.

1. INTRODUCTION

Rationalizability was independently introduced by the seminal papers of Bernheim (1984) and Pearce (1984). Soon after that, Brandenburger and Dekel (1987) defined the slightly more general concept of correlated rationalizability which allows players to hold correlated beliefs about the opponents' strategy profile. Correlated rationalizability was quickly recognized as one of the central solution concepts in non-cooperative game theory.

In fact, there are two equivalent definitions of correlated rationalizability. On the one hand, the iterative definition yields the strategy profiles that survive iterated elimination of strictly dominated strategies. On the other hand, according to the fixed point definition a strategy profile is correlated rationalizable whenever it belongs to a best response set.[1] While the two definitions are equivalent, in the sense that they induce exactly the same predictions, yet they differ in conceptual terms.

First recall the standard epistemic foundation for the iterative definition, according to which correlated rationalizability is characterized by common belief of rationality (Böge and Eisele, 1979; Tan and Werlang, 1988; Brandenburger and Dekel, 1987).[2] Notice that this epistemic characterization describes a natural reasoning process undertaken by each player separately.

On the other hand, Zambrano (2008) provided sufficient conditions for the fixed point definition of correlated rationalizability that do not involve common belief of rationality.[3] More specifically, he

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[1]A best response set is a product of justifiable strategies of each player, with the property that each justifiable strategy of each player is a best response to a (possibly correlated) belief over the opponents' justifiable strategy profiles.

[2]For an overview of this literature we refer to Perea (2012).

[3]In fact, Zambrano (2008) uses the term "knowledge" for probability-1 belief.

showed that mutual belief of (i) the payoff functions, (ii) rationality, and (iii) the fact that every player deems possible only strategy profiles that belong to the support of each player's actual conjecture, lead to everybody believing that a strategy profile in a best response set will be played. Unlike the previously-mentioned characterization of the iterative definition, these conditions do not correspond to a specific reasoning process. Instead, they can be viewed as the modeler's point of view about each player's beliefs. That is, if the modeler observes that all players jointly satisfy Zambrano's conditions, then she can infer that the players have coordinated to a best response set, and therefore a correlated rationalizable strategy profile will be played. This is for instance the case when the modeler knows that the players communicate their beliefs to each other according to some protocol, and at the steady state of this communication process Zambrano's conditions are satisfied. The idea is similar to the one behind the standard epistemic conditions for Nash equilibrium by Aumann and Brandenburger (1995).

In a recent paper, Bach and Tsakas (2012) introduced a framework for modeling pairwise epistemic conditions for a pair of players. Then, they showed that in the existence of a common prior, pairwise mutual belief of the payoff functions, pairwise mutual belief of rationality and pairwise common belief of conjectures for *only some pairs of players* suffice for a Nash equilibrium. Notably, their conditions do not require nor imply mutual belief of rationality. The conceptual implication of the previous result is that the modeler may observe the players communicate privately, and at the steady state of this communication process their beliefs satisfy conditions that are sufficient for a Nash equilibrium, implying that they manage to coordinate on a Nash equilibrium even if they do not have direct access to each other, like it often happens in social networks.

In this paper, we use this framework to provide sufficient epistemic conditions for correlated rationalizability. Our conditions are based on imposing pairwise mutual belief of payoffs, pairwise mutual belief of rationality and pairwise mutual belief of every player deeming possible only strategy profiles that belong to the support of the actual conjecture *only for some pairs of players*. Indeed, if there exists a common prior, our conditions lead to strategy profiles that satisfy the fixed point definition of correlated rationalizability. Surprisingly, our conditions do not require nor imply mutual belief in rationality.

The technical implication of our result is straightforward. Namely, our conditions are a weakening of the ones by Zambrano (2008) under a common prior.[4] This weakening becomes significant in games with a large number of players. What is more interesting is the conceptual implications of our result. On the one hand, similarly to the result of Bach and Tsakas (2012), our conditions may emerge as the steady state of a process of private communication, thus implying that the players may coordinate on a best response set even if they do not have direct access to each other. At the same time, our

[4]Unlike Aumann and Brandenburger (1995) as well as the present paper, Zambrano (2008) does not assume a common prior.

result provides additional insight on the relationship between correlated rationalizability and Nash equilibrium. To see this, observe that our conditions are weaker than the ones of Bach and Tsakas (2012, Cor. 1), which lead to Nash equilibrium, in two different aspects. Namely, compared to their result, we weaken pairwise common belief in conjectures by instead requiring pairwise mutual belief of every player deeming possible only strategy profiles that belong to the support of the actual conjecture, while at the same time we require fewer connections among the players. This may suggest that in order to reach the level of coordination needed for a Nash equilibrium – which is much higher than the level of coordination needed for the fixed point definition of correlated rationalizability – the degree of connectivity plays a crucial role. Indeed, Bach and Tsakas (2012) showed that G-pairwise mutual belief in the payoffs, G-pairwise mutual belief in rationality and G-pairwise common belief in conjectures do not in general suffice for Nash equilibrium when G is simply connected. However, it follows as a direct corollary of our result that they would suffice for correlated rationalizability.

2. PRELIMINARIES

2.1. *Normal form games*

Let $\big(I,(A_i)_{i\in I},(g_i)_{i\in I}\big)$ be game in normal form, where $I=\{1,\dots,n\}$ denotes the finite set of players with typical element i, and A_i denotes the finite set of (pure) strategies, also called actions, with typical element a_i for every player $i\in I$. As usual, define $A:=\prod_{i\in I}A_i$ with typical element $a=(a_1,\dots,a_n)$ and $A_{-i}:=\prod_{j\in I\setminus\{i\}}A_j$ with typical element $a_{-i}=(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n)$. The function $g_i : A_i\times A_{-i}\to\mathbb{R}$ denotes player i's payoff function.

A probability measure $\phi_i\in\Delta(A_{-i})$ on the set of the opponents' strategy profiles is called a conjecture of i, with $\phi_i(a_{-i})$ signifying the probability that i attributes to the opponents playing a_{-i}. Slightly abusing notation, let $\phi_i(a_j)$ denote the probability that i assigns to j playing a_j. As usual we allow for correlated beliefs, i.e., ϕ_i is not necessarily a product measure, hence the probability $\phi_i(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n)$ can differ from the product $\phi_i(a_1)\cdots\phi_i(a_{i-1})\phi_i(a_{i+1})\cdots\phi_i(a_N)$ of the marginal probabilities.[5] We say that an action a_i is a best response to ϕ_i, and write $a_i\in BR_i(\phi_i)$, whenever

$$\sum_{a_{-i}\in A_{-i}}\phi_i(a_{-i})g_i(a_i,a_{-i})\geq\sum_{a_{-i}\in A_{-i}}\phi_i(a_{-i})g_i(a_i',a_{-i})$$

for all $a_i'\in A_i$.

For each $i\in I$, consider some $C_i\subseteq A_i$. Then, we say that the rectangle $C_1\times\cdots\times C_n$ satisfies the *best response property* whenever for each $a_i\in C_i$ there exists some $\phi_i\in\Delta(C_{-i})$ with $a_i\in BR_i(\phi_i)$

[5]Intuitively, a player's belief on his opponents' strategies can be correlated, even though players choose independently from each other.

(Brandenburger and Dekel, 1987, Def. 2.1). Notice that the previous definition of the best response property differs from the one by Pearce (1984, Def. 2) in that Pearce requires that for every a_i the conjecture ϕ_i with $a_i \in BR_i(\phi_i)$ is such that the marginal distributions are independent. Throughout the present paper the term best response property refers to the definition by Brandenburger and Dekel (1987).

A strategy profile $(a_1, \dots, a_n)$ is said to be *correlated rationalizable* whenever there is some $C_1 \times \dots \times C_n$ satisfying the best response property such that $(a_1, \dots, a_n) \in C_1 \times \dots \times C_n$. This is called the *fixed point definition* of correlated rationalizability, as opposed to the alternative, yet equivalent *iterative definition* according to which a strategy profile is correlated rationalizable if and only if it survives iterated elimination of strictly dominated strategies. While the two definitions are equivalent by means of the predictions they yield, they are still different in conceptual terms.

2.2. *Epistemic Models*

Recall the standard epistemic model of Aumann and Brandenburger (1995), also used in Zambrano (2008): Let S_i be a finite set of types[6] for each player i, with typical element s_i. As usual, let $S := S_1 \times \dots \times S_n$ and $S_{-i} := S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_n$. An element $s = (s_1, \dots, s_n)$ of S is called state of the world, or simply state, while every subset of S is called an event. The event $[s_i] := \{s \in S : \mathrm{Proj}_{S_i} s = s_i\}$ contains all states at which i's type is s_i. Each type $s_i \in S_i$ is associated with a probability measure over S_{-i}, called s_i's theory, which induces s_i's distribution $p(\cdot; s_i) \in \Delta(S)$ over the state space by attaching to each $E \subseteq S$ the probability that s_i's theory assigns to $\{s_{-i} \in S_{-i} : (s_i, s_{-i}) \in E\}$. The extension from s_i's theory to $p(\cdot; s_i)$ is unique since we assume $p([s_i]; s_i) = 1$ and also that $\mathrm{marg}_{S_{-i}} p(\cdot; s_i)$ coincides with s_i's theory (Aumann and Brandenburger, 1995). Intuitively, $p(\cdot; s_i)$ denotes i's conditional beliefs over the state space given the type s_i. A probability measure $P_i \in \Delta(S)$ is called player i's prior, if for all $s_i \in S_i$ the conditional distribution of P_i given s_i coincides with $p(\cdot; s_i)$. If $P_i = P$ for every $i \in I$, we say that P is a common prior.

Belief is formalized in terms of events: the set of states where agent i believes in $E \subseteq S$ is defined as

$$B_i(E) := \{s \in S : p(E; s_i) = 1\}.$$

Then, it is said that i believes in E at s, whenever $s \in B_i(E)$. Note that Aumann and Brandenburger (1995) actually use the term knowledge for probability-1 belief.

An event is mutually believed if everyone believes it. Formally, $E \subseteq S$ is mutually believed at s,

[6] The finiteness assumption is without loss of generality, as our results can be generalized to arbitrary measurable types spaces.

whenever $s \in B(E)$, where

$$B(E) := \bigcap_{i \in I} B_i(E).$$

For every player $i \in I$ an action function $\mathbf{a}_i : S \to A_i$ specifies his action at each state, and it is assumed to be S_i-measurable, i.e., $\mathbf{a}_i(s) = \mathbf{a}_i(s')$ if $\{s, s'\} \subseteq [s_i]$, implying that i attaches probability 1 to his actual strategy. The event $[a_i] := \{s \in S : \mathbf{a}_i(s) = a_i\}$ contains the states at which agent i plays a_i, and $[a_{-i}] := [a_1] \cap \cdots \cap [a_{i-1}] \cap [a_{i+1}] \cap \cdots \cap [a_n]$.

The function $\boldsymbol{\phi}_i : S \to \Delta(A_{-i})$ specifies i's conjecture at every state, and is defined by

$$\boldsymbol{\phi}_i(s)\big([a_{-i}]\big) := p\big([a_{-i}]; s_i\big)$$

for each $a_{-i} \in A_{-i}$, and is by construction S_i-measurable, i.e., $\boldsymbol{\phi}_i(s) = \boldsymbol{\phi}_i(s')$ if $\{s, s'\} \subseteq [s_i]$, implying that i assign probability 1 to his actual conjecture. We define the events $[\phi_i] := \{s \in S : \boldsymbol{\phi}_i(s) = \phi_i\}$ and $[\phi_1, \dots, \phi_n] := [\phi_1] \cap \cdots \cap [\phi_n]$. Throughout the paper, for an arbitrary $C_{-i} \subseteq A_{-i}$, let

$$D_i(C_{-i}) := \big\{\ s \in S : \mathrm{Supp}\big(\boldsymbol{\phi}_i(s)\big) \subseteq C_{-i}\ \big\}$$

denote the states where player i deems possible only opponents' strategy profiles belonging to C_{-i}.

Finally, $\mathbf{g}_i : S \times A \to \mathbb{R}$ specifies i's *payoff function* at each state of the world, and it is assumed that $\mathbf{g}_i$ is also S_i-measurable, i.e. $\mathbf{g}_i(s, a) = \mathbf{g}_i(s', a)$ if $\{s, s'\} \subseteq [s_i]$ for all $a \in A$, which implies that i attaches probability 1 to his actual payoff function. For some fixed $g_i : A \to \mathbb{R}$, let $[g_i] := \{s \in S : \mathbf{g}_i(s, a) = g_i(a),$ for all $a \in A\}$ denote the states where i's payoff function is g_i. Then, we also define $[g_1, \dots, g_n] := [g_1] \cap \cdots \cap [g_n]$. A game is said to be of complete information if there exists $(g_1, \dots, g_n)$ such that $[g_1, \dots, g_n] = S$.

Furthermore, player i is rational at some state s, whenever he maximizes his expected payoff at this state given his conjecture and payoff function. That is,

$$R_i := \big\{\ s \in S : \mathbf{a}_i(s) \in BR_i\big(\boldsymbol{\phi}_i(s)\big)\ \big\}$$

denotes the event that i is rational.

2.3. *Epistemic conditions for correlated rationalizability*

According to the standard epistemic characterization of correlated rationalizability, a strategy profile survives iterated elimination of strictly dominated strategies – and therefore is correlated rationalizable – if and only if it can be rationally played under common belief in rationality (Böge and Eisele, 1979; Brandenburger and Dekel, 1987; Tan and Werlang, 1988). Notice that this epistemic characterization describes a natural reasoning process undertaken by each player separately. Namely, if a player believes that everybody is rational, then he also believes that no player will choose a strictly dominated strategy, and therefore these strategies can be eliminated. Moreover, if he believes that everybody believes that everybody is rational, then he also believes that every player has eliminated the strictly dominated strategies, and so on. Thus, common belief in rationality leads the player to rule out the strategy profiles that do not satisfy the conditions of the iterative definition of correlated rationalizability.

In a recent paper, Zambrano (2008) provided sufficient epistemic conditions that do not involve common belief in rationality. More specifically, he showed that if at some state it is mutually believed (i) which are the payoff functions of each player, (ii) that every player is rational, and (iii) that each player deems possible only strategy profiles that receive positive probability by actual conjectures, then a correlated rationalizable strategy is played. Formally, his result is stated as follows.

THEOREM 1 (Zambrano, 2008) *Let* $\big(I,(A_i)_{i\in I},(g_i)_{i\in I}\big)$ *be a normal form game. Suppose that there exists a state* $s\in S$ *such that* $s\in B\big([g_1,\dots,g_n]\big)\cap B(R_1\cap\cdots\cap R_n)\cap B\Big(\bigcap_{i\in N}D_i\big(\operatorname{Supp}(\boldsymbol{\phi}_i(s))\big)\Big)$. *Then,*

$$\Big(\bigcup_{j\neq 1}\operatorname{Proj}_{A_1}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)\Big)\times\cdots\times\Big(\bigcup_{j\neq n}\operatorname{Proj}_{A_n}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)\Big)$$

satisfies the best response property, and therefore $\big(\mathbf{a}_1(s),\dots,\mathbf{a}_n(s)\big)$ *is correlated rationalizable.*

Unlike common belief in rationality, these epistemic conditions are not directly related with a specific reasoning process employed by each player. Instead they can be thought as the modeler's point of view regarding the belief hierarchies of all players jointly. For instance, this is the case when the modeler knows that the players communicate according to a certain protocol, and at the steady state of this process the beliefs of each player satisfy the conditions of the previous result. Finally, notice that these epistemic restrictions lead to strategy profiles that satisfy the conditions of the fixed point definition of correlated rationalizability.

3. PAIRWISE MUTUAL BELIEF

Recall the notion of pairwise mutual belief, first introduced in Bach and Tsakas (2012): Let $E\subseteq S$ be some event and $i,j\in I$ be two players. We say that E is *pairwise mutual belief* between i and j whenever they both believe E. Formally, pairwise mutual belief of E between i and j is denoted by the event

$$B_{i,j}(E) := B_i(E)\cap B_j(E).$$

In contrast to the standard notion of mutual belief, our pairwise epistemic operator describes mutual belief only locally for pairs of agents, postulating the existence of exclusively binary relations of epistemic relevance. Formally, we represent a set of such binary relations by means of an undirected graph $G = (I, \mathcal{E})$, where the set of vertices I denotes the set of players, and the set of edges $\mathcal{E}$ describe binary symmetric relations $(i, j) \in I \times I$ between pairs of players.

The graph G does neither enrich the epistemic model nor add any additional structure to the game whatsoever, but only provides a formal framework for expressing pairwise local conditions of mutual belief. In general, the connectedness of two agents by an edge admits two complementary interpretations. On the one hand, an edge may be of *epistemic character*, while on the other hand, G may be interpreted as a *social network*.[7]

Next, some graph theoretic notions are recalled. A sequence $(i_k)_{k=1}^{m}$ of players is a *path* whenever $(i_k, i_{k+1}) \in \mathcal{E}$ for all $k \in \{1, \ldots, m-1\}$, i.e. in a path every two consecutive players are linked by an edge. A graph G is called *connected* if it contains a path $(i_k)_{k=1}^{m}$ such that for every $i \in I$ there is some $k \in \{1, \ldots, m\}$ with $i_k = i$. In addition, G is *complete*, if $(i, j) \in \mathcal{E}$ for all $i, j \in I$.

Specific pairwise mutual belief conditions are now introduced.

Definition 1 Let $\big(I, (A_i)_{i \in I}, (g_i)_{i \in I}\big)$ be a game, G be an undirected graph and s be a state.

- *Payoffs are G-pairwise mutual belief at s* whenever $s \in B_{i,j}\big([g_i] \cap [g_j]\big)$ for all $(i, j) \in \mathcal{E}$.
- *Rationality is G-pairwise mutual belief at s* whenever $s \in B_{i,j}(R_i \cap R_j)$ for all $(i, j) \in \mathcal{E}$.
- *It is G-pairwise mutual belief at s that players deem possible only strategy profiles belonging to the actual support of their conjectures* whenever $s \in B_{i,j}\Big(D_i\big(\mathrm{Supp}(\boldsymbol{\phi}_i(s))\big) \cap D_j\big(\mathrm{Supp}(\boldsymbol{\phi}_j(s))\big)\Big)$ for all $(i, j) \in \mathcal{E}$.

Note that henceforth an edge between two agents i and j in a graph G signifies that i and j entertain (i) pairwise mutual belief of the payoffs, (ii) pairwise mutual belief of rationality, and (iii) pairwise mutual belief of everybody deeming possible only strategy profiles belonging to the actual support of their conjectures.

It is straightforward verifying that the epistemic conditions introduced in Definition 1 are weaker than the ones used by Zambrano (2008). Observe that in our context mutual belief coincides with G-pairwise mutual belief whenever G is complete.

The following example illustrates the new concepts of G-pairwise mutual belief and also relates them to the standard notions of mutual belief as used by Zambrano (2008).

[7]For an extensive discussion on the different interpretations see Bach and Tsakas (2012).

Example 1 Consider the symmetric normal form game $\big(I, (A_i)_{i\in I}, (g_i)_{i\in I}\big)$, where $I = \{\text{Ann } (A), \text{Bob } (B), \text{Carol } (C)\}$ is the set of players, and $A_i = \{h, \ell\}$ the finite set of strategies of each $i \in I$. The payoff functions of Ann and Bob are independent of the opponents' strategy profile, i.e., for each $i \in \{A, B\}$,

$$U_i(a_i, a_{-i}) = \begin{cases} 1 & \text{if } a_i = h \\ 0 & \text{if } a_i = \ell \end{cases}$$

for every $a_{-i} \in A_{-i}$. On the other hand, Carol's payoff function is given by

$$U_C(a_A, a_B, a_C) = \begin{cases} 1 & \text{if } (a_A, a_B, a_C) = (h, h, h) \\ 2 & \text{if } (a_A, a_B, a_C) = (\ell, \ell, \ell) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the only correlated rationalizable strategy profile is (h, h, h): Playing ℓ is strictly dominated for Ann and Bob, and therefore they both eliminate it. Then, at the second round of elimination, Carol wipes out ℓ, as $U_C(h, h, h) > U_C(\ell, h, h)$.

Now, consider the type spaces:

$$\begin{aligned} S_A &= \{s_A^1(h), s_A^2(\ell)\}, \\ S_B &= \{s_B^1(h), s_B^2(h), s_B^3(\ell)\}, \\ S_C &= \{s_C^1(h), s_C^2(h)\}, \end{aligned}$$

with the action in parenthesis denoting the respective player i's action at every state given by the function $\mathbf{a}_i$. Moreover, suppose that the players have a common prior

$$P \text{ uniformly distributed over } \big\{ \, (s_A^1, s_B^1, s_C^1), \ (s_A^1, s_B^2, s_C^2), \ (s_A^2, s_B^3, s_C^2) \, \big\} \, .$$

For instance, if Ann's type is s_A^1, then she attaches probability $\frac{1}{2}$ to (s_A^1, s_B^1, s_C^1) and $\frac{1}{2}$ to (s_A^1, s_B^2, s_C^2). Let $G = (I, \mathcal{E})$ be a connected graph such that

$$\begin{aligned} I &= \{\text{Ann, Bob, Carol}\}, \\ \mathcal{E} &= \{(\text{Ann, Bob}), (\text{Bob, Carol})\}. \end{aligned}$$

Firstly, observe that it is G-pairwise mutual belief at (s_A^1, s_B^1, s_C^1) that players deem possible only strategy profiles belonging to the support of the actual conjectures. More specifically, notice that the states where the players deem possible only strategy profiles that belong to the supports of the conjectures at (s_A^1, s_B^1, s_C^1) are

$$\begin{aligned} D_A\big(\operatorname{Supp}(\boldsymbol{\phi}_A(s_A^1, s_B^1, s_C^1))\big) &= \big\{ \, s \in S : \operatorname{Supp}\big(\boldsymbol{\phi}_A(s)\big) \subseteq \operatorname{Supp}\big(\boldsymbol{\phi}_A(s_A^1, s_B^1, s_C^1)\big) \, \big\} \\ &= \big\{ \, s \in S : \operatorname{Supp}\big(\boldsymbol{\phi}_A(s)\big) \subseteq \{(h, h)\} \, \big\} \\ &= \{(s_A^1, s_B^1, s_C^1), (s_A^1, s_B^2, s_C^2)\}, \end{aligned}$$

and likewise

$$\begin{aligned} D_B\big(\operatorname{Supp}(\boldsymbol{\phi}_B(s_A^1, s_B^1, s_C^1))\big) &= \{(s_A^1, s_B^1, s_C^1), (s_A^1, s_B^2, s_C^2)\}, \\ D_C\big(\operatorname{Supp}(\boldsymbol{\phi}_C(s_A^1, s_B^1, s_C^1))\big) &= \{(s_A^1, s_B^1, s_C^1)\}. \end{aligned}$$

Then, it is straightforward verifying that

$$\begin{aligned} (s_A^1, s_B^1, s_C^1) &\in B_{A,B}(\{(s_A^1, s_B^1, s_C^1), (s_A^1, s_B^2, s_C^2)\}) \cap B_{B,C}(\{(s_A^1, s_B^1, s_C^1)\}) \\ &= B_{A,B}\Big(D_A\big(\operatorname{Supp}(\boldsymbol{\phi}_A(s_A^1, s_B^1, s_C^1))\big) \cap D_B\big(\operatorname{Supp}(\boldsymbol{\phi}_B(s_A^1, s_B^1, s_C^1))\big)\Big) \\ &\cap B_{B,C}\Big(D_B\big(\operatorname{Supp}(\boldsymbol{\phi}_B(s_A^1, s_B^1, s_C^1))\big) \cap D_C\big(\operatorname{Supp}(\boldsymbol{\phi}_C(s_A^1, s_B^1, s_C^1))\big)\Big). \end{aligned}$$

However, observe that it is not mutually believed at (s_A^1, s_B^1, s_C^1) that players deem possible only strategy profiles that receive positive probability by their actual conjecture at (s_A^1, s_B^1, s_C^1), since it is the case that $(s_A^1, s_B^1, s_C^1) \notin B_A\Big(D_c\big(\operatorname{Supp}(\boldsymbol{\phi}_C(s_A^1, s_B^1, s_C^1))\big)\Big)$, implying that the second condition of Theorem 1 is violated.

Secondly, note that rationality is G-pairwise mutual belief at (s_A^1, s_B^1, s_C^1). However, it is not mutually believed at (s_A^1, s_B^1, s_C^1) that everyone is rational. Indeed, Ann does not believe at (s_A^1, s_B^1, s_C^1) that Carol is rational, as at (s_A^1, s_B^2, s_C^2) Carol's unique best response to her conjecture is to play ℓ rather than h, implying that the first condition of Theorem 1 is not satisfied either.

Finally, observe that

$$\prod_{i \in I}\Big(\bigcup_{j \neq i} \operatorname{Proj}_{A_i} \operatorname{Supp}\big(\boldsymbol{\phi}_j(s_A^1, s_B^1, s_C^1)\big)\Big) = \{h\} \times \{h\} \times \{h\}$$

is the unique rectangle satisfying the best response property, and therefore $\big(\mathbf{a}_A(s_A^1), \mathbf{a}_B(s_B^1), \mathbf{a}_C(s_C^1)\big) = (h, h, h)$ is a correlated rationalizable strategy profile. $\triangleleft$

In the preceding example, both the two central elements of Zambrano's sufficient conditions for correlated rationalizability are violated, and yet the conclusion of his theorem does hold. On the basis of this observation, the natural question then arises, whether there exists a general relation between our G-pairwise mutual belief conditions of Definition 1 on the one hand, and correlated rationalizability on the other hand. The answer to this question is affirmative in the existence of a common prior, as we show in the next section.

4. PAIRWISE MUTUAL BELIEF AND EPISTEMIC CONDITIONS FOR RATIONALIZABILITY

The following result provides sufficient conditions for correlated rationalizability by means of pairwise mutual belief. More specifically, it is shown that G-pairwise mutual belief of the payoff functions, G-pairwise mutual belief of rationality and G-pairwise mutual belief of everybody deeming possible only strategy profiles belonging to the actual support of their conjectures already suffice for correlated rationalizability, if there is a common prior and G is connected.

Theorem 2 *Let $\big(I,(A_i)_{i\in I},(g_i)_{i\in I}\big)$ be a normal form game and $G=(I,\mathcal{E})$ be a connected graph. Suppose that there exists a common prior attaching positive probability to a state $s\in S$ such that $s\in B_{i,j}\big([g_i]\cap[g_j]\big)\cap B_{i,j}(R_i\cap R_j)\cap B_{i,j}\Big(D_i\big(\operatorname{Supp}(\boldsymbol{\phi}_i(s))\cap D_j\big(\operatorname{Supp}(\boldsymbol{\phi}_j(s))\big)\big)\Big)$ for all $(i,j)\in\mathcal{E}$. Then,*

$$\Big(\bigcup_{j\neq 1}\operatorname{Proj}_{A_1}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)\Big)\times\cdots\times\Big(\bigcup_{j\neq n}\operatorname{Proj}_{A_n}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)\Big) \tag{1}$$

satisfies the best response property, and therefore $\big(\mathbf{a}_1(s),\ldots,\mathbf{a}_n(s)\big)$ is correlated rationalizable.

Proof: Consider an arbitrary $(i,j)\in\mathcal{E}$ and let $k\in I\backslash\{i,j\}$. First, we show that $\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_i(s)\big)\subseteq\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)$. By definition, it follows from $s\in B_i\Big(D_j\big(\operatorname{Supp}(\boldsymbol{\phi}_j(s))\big)\Big)$ that $\operatorname{Supp}(\boldsymbol{\phi}_j(s'))\subseteq\operatorname{Supp}(\boldsymbol{\phi}_j(s))$ for every $s'\in S$ with $p(\{s'\};s_i)>0$, and therefore it also follows that

$$\operatorname{Proj}_{A_k}\operatorname{Supp}(\boldsymbol{\phi}_j(s'))\subseteq\operatorname{Proj}_{A_k}\operatorname{Supp}(\boldsymbol{\phi}_j(s)) \tag{2}$$

for every $s'\in S$ with $p(\{s'\};s_i)>0$. Now, consider some $a_k\in\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_i(s)\big)$. Then, there is some $s'\in S$ with $p(\{s'\};s_i)>0$, such that $\mathbf{a}_k(s')=a_k$. Since the common prior P attaches positive probability to s – and therefore to $[s_i]$ – it is the case that $P(\{s'\}\cap[s_i])=p(\{s'\};s_i)\cdot P([s_i])>0$. Hence, it is also the case that $P(\{s'\})>0$. Then, it follows from $s'\in[a_k]\cap[s'_j]$ that $\boldsymbol{\phi}_j(s')(a_k)=P\big([a_k]\cap[s'_j]\big)/P\big([s'_j]\big)>0$. Therefore, $a_k\in\operatorname{Proj}_{A_k}\operatorname{Supp}(\boldsymbol{\phi}_j(s'))$, thus implying by Eq. (2) that $a_k\in\operatorname{Proj}_{A_k}\operatorname{Supp}(\boldsymbol{\phi}_j(s))$. Likewise, we prove that $\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)\subseteq\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_i(s)\big)$. Therefore, we obtain

$$\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)=\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_i(s)\big). \tag{3}$$

Now, we consider an arbitrary $i\in I$ and we show that for every $a_i\in\bigcup_{k\neq i}\operatorname{Proj}_{A_i}\operatorname{Supp}\big(\boldsymbol{\phi}_k(s)\big)$ there is some conjecture

$$\phi_i\in\Delta\bigg(\prod_{k\neq i}\Big(\bigcup_{\ell\neq k}\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_\ell(s)\big)\Big)\bigg)$$

such that $a_i\in BR_i(\phi_i)$, which then implies that the rectangle in (1) satisfies the best response property. Since $a_i\in\bigcup_{k\neq i}\operatorname{Proj}_{A_i}\operatorname{Supp}\big(\boldsymbol{\phi}_k(s)\big)$, there is some $k\neq i$ such that $a_i\in\operatorname{Proj}_{A_i}\operatorname{Supp}\big(\boldsymbol{\phi}_k(s)\big)$. Since G is connected there exists a path connecting i with k, and let j be the vertex in this path such that $(i,j)\in\mathcal{E}$. Notice that j coincides with k if it is the case that $(i,k)\in\mathcal{E}$. Then, it follows from Eq. (3) that $a_i\in\operatorname{Proj}_{A_i}\operatorname{Supp}\big(\boldsymbol{\phi}_j(s)\big)$. Therefore, there is some $s''\in S$ with $p(\{s''\};s_j)>0$ such that $\mathbf{a}_i(s'')=a_i$. Moreover, it follows from $s\in B_j(R_i)\cap B_j\big([g_i]\big)$ that $\mathbf{a}_i(s'')\in BR_i\big(\boldsymbol{\phi}_i(s'')\big)$. Furthermore, it follows from $s\in B_j\Big(D_i\big(\operatorname{Supp}(\boldsymbol{\phi}_i(s))\big)\Big)$ that $\operatorname{Supp}(\boldsymbol{\phi}_i(s''))\subseteq\operatorname{Supp}(\boldsymbol{\phi}_i(s))$. Finally, notice that

$$\begin{aligned}\operatorname{Supp}(\boldsymbol{\phi}_i(s)) &\subseteq \prod_{k\neq i}\operatorname{Proj}_{A_k}\operatorname{Supp}(\boldsymbol{\phi}_i(s))\\ &\subseteq \prod_{k\neq i}\Big(\bigcup_{\ell\neq k}\operatorname{Proj}_{A_k}\operatorname{Supp}\big(\boldsymbol{\phi}_\ell(s)\big)\Big)\end{aligned}$$

thus implying that $\phi_i(s'') \in \Delta\Big(\prod_{k\neq i}\Big(\bigcup_{\ell\neq k} \mathrm{Proj}_{A_k} \mathrm{Supp}\big(\phi_\ell(s)\big)\Big)\Big)$, which completes the proof. *Q.E.D.*

The technical contribution of the previous result is straightforward. Firstly, it weakens Zambrano's epistemic conditions for correlated rationalizability when beliefs are derived from a common prior. Secondly, it provides sufficient epistemic conditions for correlated rationalizability without even mutual belief of rationality.

Our previous result has also several conceptual implications. Firstly, recall the interpretation of Zambrano's conditions from Section 2.3. According to this interpretation, the players communicate with each other, and they eventually reach a steady state that satisfies the assumptions of Theorem 1. In the context of our result such communication would only need to take place between pairs of connected players. Of course, in this paper we are interested in the strategy profile that is played once convergence has already taken place, rather than in the communication process that has led to this steady state. Providing sufficient conditions on the communication mechanism that would lead to the specific epistemic conditions remains an open question for future research.

Secondly, in a recent paper Bach and Tsakas (2012) provided sufficient pairwise epistemic conditions for Nash equilibrium. Namely, they showed that if there exists a common prior attaching positive probability to some state at which the payoff functions and rationality are G-pairwise mutual belief and conjectures are G-pairwise common belief, the (common) marginal conjectures will form a Nash equilibrium as long as the graph is biconnected[8] (Bach and Tsakas, 2012, Cor. 1). Observe that these epistemic conditions for Nash equilibrium are stronger than the ones of Theorem 2 in two different dimensions. On the one hand, they impose pairwise common belief of conjectures instead of pairwise mutual belief of the supports of the conjectures. At the same time, they require a higher degree of connectivity as biconnected graphs are always connected, but not necessarily the other way around. In fact, Bach and Tsakas (2012) showed that G-pairwise mutual belief in the payoffs, G-pairwise mutual belief in rationality and G-pairwise common belief in conjectures do not in general suffice for Nash equilibrium when G is simply connected. However, notice that the same G-pairwise epistemic conditions suffice for correlated rationalizability even if the graph is simply connected and not biconnected. This fact may suggest that in order to reach the level of coordination needed for a Nash equilibrium – which is much higher than the level of coordination needed for correlated rationalizability – we must have a larger degree of connectivity among the players.

[8]A graph is biconnected if it connected and also has the property that after removing an arbitrary node/player it remains connected.

5. DISCUSSION

Tightness of the main result. Notice that the assumption about the graph being connected is crucial for Theorem 2. To see this, recall the game from Example 1, and suppose that there is a state s where both Ann and Bob play h while Carol plays ℓ. Moreover, assume that this state is commonly believed. Then, observe that all our pairwise conditions of Theorem 2 are satisfied between Ann and Bob, and still the conclusion of the result fails. The reason is that pairwise mutual belief in rationality is not satisfied for any pair containing Carol, since both Ann and Bob put probability 1 to Carol being irrational, implying that G is not connected.

Our common prior assumption is also crucial for Theorem 2, as shown in the next example.

EXAMPLE 2 Recall the game introduced in Example 1, and consider the type space

$$\begin{aligned} S_A &= \{s_A^1(h)\}, \\ S_B &= \{s_B^1(h)\}, \\ S_C &= \{s_C^1(h), s_C^2(\ell)\}, \end{aligned}$$

with the action in parenthesis denoting the respective player i's action at every state given by the function $\mathbf{a}_i$. Suppose that s_A^1 assigns probability 1/2 to each state in $S = \{(s_A^1, s_B^1, s_C^1), (s_A^1, s_B^1, s_C^2)\}$, while s_B^1 puts probability 1 to (s_A^1, s_B^1, s_C^1). Obviously, Carol is always certain of the actual state, as the only thing that distinguishes the two states is her own type. Now, let $G = (I, \mathcal{E})$ be a connected graph such that

$$\begin{aligned} I &= \{\text{Ann, Bob, Carol}\}, \\ \mathcal{E} &= \{(\text{Ann, Bob}), (\text{Bob, Carol})\}, \end{aligned}$$

and observe that all the G-pairwise epistemic conditions of Theorem 2 hold at (s_A^1, s_B^1, s_C^2), and still the strategy profile (h, h, ℓ) played at this state is not correlated rationalizable. The reason is that there is no common prior from which these beliefs could have been derived. $\triangleleft$

From the previous discussion it becomes apparent that our result is tight in the sense that none of the two assumptions can be omitted.

Converse of the main result. The sufficient epistemic conditions imposed by our main result are not always necessary, and in this sense our result does not provide a characterization of the set of correlated rationalizable strategy profiles. What can be shown instead is that for any given connected graph G and for every correlated rationalizable strategy profiles, there exists some epistemic model satisfying our G-pairwise epistemic conditions, with the property that the best response set from our theorem contains this strategy profile. Hence, our conditions do not lead to a refinement of correlated rationalizability.

We are not going to provide a formal proof of this claim, but instead sketch the main argument behind it. Consider the set of correlated rationalizable strategy profiles $C_1 \times \cdots \times C_n$, and let $(a_1, \ldots, a_n)$ be a strategy profiles such no a_i is weakly dominated in $C_1 \times \cdots \times C_n$, implying that there exists a full support belief $\phi_i \in \Delta\big(\prod_{j \neq i} C_j\big)$ such that $a_i \in BR_i(\phi_i)$. Observe that such a strategy profiles always exists. Suppose that $\mathbf{a}_i(s) = a_i$ for all $i \in I$, and also assume that for any pair $(i, j) \in I \times I$ it is the case that $\operatorname{Supp} p(\cdot; s_i) \cap \operatorname{Supp} p(\cdot; s_j) = \{s\}$ and that $\operatorname{Supp}\big(\boldsymbol{\phi}_i(s)\big) = \prod_{j \neq i} C_j$. Now consider a prior that assigns positive probability to s. Furthermore, for each pair $(i, j) \in I \times I$, suppose that our pairwise conditions of Theorem 2 are satisfied, implying that we are considering the complete graph. Notice that such an epistemic model exists, since the only state that two different players deem possible at s is s itself. Then, observe that the best response of Theorem 2 is $C_1 \times \cdots \times C_n$. Finally, notice that we could have instead used any connected graph for this specific example.

Infinite type spaces. For presentation purposes, we have restricted our analysis to finite epistemic models. It is straightforward to extend our result to infinite type spaces. To see this, observe that the finiteness of the state space is only used in the proof of Theorem 2 to show that $a_k \in \operatorname{Proj}_{A_k} \operatorname{Supp}\big(\boldsymbol{\phi}_i(s)\big)$ implies the existence of a state $s' \in S$ such that $p(\{s'\}; s_i) > 0$ and $\mathbf{a}_k(s') = a_k$. In an infinite measurable state space, we would instead use the fact that $a_k \in \operatorname{Proj}_{A_k} \operatorname{Supp}\big(\boldsymbol{\phi}_i(s)\big)$ implies the existence of a Borel subset $E \subseteq S$ such that $p(E; s_i) > 0$ and also $\mathbf{a}_k(s') = a_k$ for all $s' \in E$, which is obviously true.

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