Universally rational belief hierarchies[☆]

Elias Tsakas

Department of Economics, Maastricht University

Abstract

In a recent paper, Tsakas (2012) introduced the notion of rational beliefs. These are Borel probability measures that assign a rational probability to every Borel event. Then, he constructed the corresponding Harsanyi type space model that represents the rational belief hierarchies. As he showed, there are rational types that are associated with a non-rational probability measure over the product of the underlying space of uncertainty and the opponent's types. In this paper, we define the universally rational belief hierarchies, as those that do not exhibit this property. Then, we characterize them in terms of a natural restriction imposed directly on the belief hierarchies.

Keywords: Epistemic game theory, rational numbers, belief hierarchies, type spaces.

1. Introduction

A belief hierarchy is a description of an agent's beliefs about some fundamental space of uncertainty, her beliefs about everybody else's beliefs, and so on. During the past few decades, belief hierarchies have been often used to analyze games with incomplete information (Harsanyi, 1967-68), as well as in order to provide epistemic characterizations for several standard solution concepts, such as rationalizability (Brandenburger and Dekel, 1987; Tan and Werlang, 1988), Nash equilibrium (Aumann and Brandenburger, 1995), and correlated equilibrium (Aumann, 1987).[1]

Belief hierarchies are in general complex objects, consisting of an infinite regression of probability measures. This makes them hard to handle and sometimes even to describe, especially when it comes to high order beliefs. Having recognized this difficulty, Harsanyi (1967-68) proposed an indirect way for representing belief hierarchies, known as the type space model. Formally, Harsanyi's model consists of a set of *types* for each agent and a continuous mapping from each type to the corresponding *beliefs* over the product of the fundamental space of uncertainty and the opponent's type space. This structure induces

[1]For an overview of the epistemic game theory literature we refer to the textbook by Perea (2012) or the review article by Brandenburger (2008).

a belief hierarchy for every type, thus reducing the infinite-dimensional regression of beliefs to a single-dimensional type. Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Mertens et al. (1994) completed the analysis by showing the existence of the universal type space, which represents all belief hierarchies satisfying some standard coherency properties.

In a recent paper Tsakas (2012) restricted attention to probabilistic beliefs that can take only rational values, e.g., he ruled out beliefs of the form "*E occurs with probability* $\sqrt{2}/2$". Such beliefs are modeled by Borel probability measures that attach a rational number to every Borel event. These Borel probability measures are called *rational*.

Considering agents who form rational beliefs over some underlying space of uncertainty Θ, like Tsakas (2012) did, does not necessarily restrict the language they use to describe their beliefs, i.e., rational beliefs are still within Harsanyi's framework which models the agents' language with the Borel σ-algebra of events in $\Delta(\Theta)$.[2] As a consequence, agents understand what it means to assign probability $\sqrt{2}/2$ to a Borel event $E \subseteq \Theta$, as the latter corresponds to the event $\{\mu \in \Delta(\Theta) : \mu(E) = \sqrt{2}/2\}$ which is Borel in $\Delta(\Theta)$. That is, in this framework, despite the fact that agents can express subjective beliefs that use non-rational probabilities, they refrain from actually doing so. In fact, recent studies indicate that human subjects find non-rational numbers very complex, often showing a tendency to rely on decimal approximations which they find more intuitive (Sirotic and Zazkis, 2007a,b).

Subsequently, Tsakas (2012) constructed a Harsanyi type space representation of rational belief hierarchies. However, as his main result shows, there exist some rational types which are associated with *non*-rational probability measures over the product of the fundamental space of uncertainty and the opponent's rational type space. In other words, there is some Borel event in this product space to which this rational type attaches a non-rational probability even though every order of her belief hierarchy involves only rational probabilities. The aim of this paper is to identify and characterize the rational types that do not exhibit this property, i.e., to provide conditions on the belief hierarchies so that the associated probability measure over the product of the underlying space of uncertainty and the opponent's rational types to assign rational probabilities to all Borel events. We call these types *universally rational*.

In order to do so, we first introduce the notion of N-rational probability measures. Accordingly, for a fixed finite subset N of the natural numbers, a rational probability measure is N-rational whenever it attaches to every Borel event a probability that can be written as fraction with the denominator belonging to N. For instance, if $N = \{100\}$, an N-rational measure would describe the beliefs of an agent whose subjective uncertainty is expressed in terms of percentages. Then, we say that a rational belief hierarchy is N-rational if all orders of beliefs are described by an N-rational probability measure. Notice, that we do not require the agent to believe that the opponent's beliefs are also N-rational, e.g., she may reason in terms of percentages,

---

[2]The standard syntactic models of logic typically assume that the language that describes the agents' beliefs is finitely generated by sentences of the form "*E occurs with probability at least p*" where p is a rational number (Fagin and Halpern, 1994; Aumann, 1999; Heifetz and Mongin, 2001; Zhou, 2010). The latter induces an algebra of events in $\Delta(\Theta)$, which is obviously coarser than the Borel σ-algebra that we consider here. To see this note that the Borel σ-algebra is the one generated by the topology of weak convergence, and is countably – rather than finitely – generated by events of the form "*E occurs with probability at least p*".

and still attach positive probability to her opponent believing some event E with probability 1/3. As it turns out universally rational types are characterized by means of N-rational beliefs. More specifically, as our main result shows (Theorem 1) a belief hierarchy is universally rational if and only if it is N-rational for some finite $N \subseteq \mathbb{N}$.

The paper is structured as follows: Section 2 recalls the framework of rational belief hierarchies; Section 3 introduces the concept of N-rational beliefs; Section 4 contains the results of the paper; All proofs are relegated to the Appendix.

2. Rational beliefs hierarchies

In this section, we recall the framework for modeling rational beliefs, introduced by Tsakas (2012).

2.1. Rational probability measures

Let X be a Polish space[3], together with the Borel σ-algebra, $\mathcal{B}$. As usual, $\Delta(X)$ denotes the space of probability measures on $(X, \mathcal{B})$, endowed with the topology of weak convergence.[4] For each $\mu \in \Delta(X)$, let supp(μ) denote the support, i.e., those $x \in X$ with the property that every open set containing x receives positive probability by μ.[5]

Consider the Borel probability measures that assign to every Borel event a rational number.

Definition 1. We define the set of *rational probability measures* by

$$\Delta^{\mathbb{Q}}(X) := \left\{ \mu \in \Delta(X) \ : \ \mu(B) \in \mathbb{Q}, \forall B \in \mathcal{B} \right\}. \tag{1}$$

We use rational probability measures to model an agent who does not hold beliefs of the form "*E occurs with probability* $\sqrt{2}/2$". Observe that the agent's language contains all events in the Borel σ-algebra generated by the topology of weak convergence, implying that she understands what it means to put probability $\sqrt{2}/2$ to E, as the latter is countably generated by events of the form $\{\mu \in \Delta(X) : \mu(E) \geq p\}$, where p is rational in $[0, 1]$. Obviously, this language is richer than the one used in logic, where only finitely generated sentences are expressible, and therefore the agent does not even understand the meaning of the sentence "*E occurs with probability* $\sqrt{2}/2$". In either case, we assume that the agent never uses such complex beliefs. The following result provides a useful, yet surprising property of rational probability measures.

Proposition 1 (Tsakas, 2012). *Every* $\mu \in \Delta^{\mathbb{Q}}(X)$ *has a finite support.*

---

[3]A topological space is called Polish whenever it is separable and completely metrizable. Examples of Polish spaces include countable sets endowed with the discrete topology and $\mathbb{R}^n$ together with the usual topology. Closed subsets of Polish spaces endowed with the relative topology are Polish. The countable product of Polish spaces, together with the product topology, is also Polish.

[4]The topology of weak convergence, which is usually denoted by w^*, is the coarsest topology that makes the mapping $\mu \mapsto \int f d\mu$ continuous, for every bounded and continuous real-valued function, f. If X is Polish, then $\Delta(X)$ endowed with the topology of weak convergence is also Polish. For further properties of w^*, we refer to Aliprantis and Border (1994, Ch. 15).

[5]If X is separable and metrizable, the support is unique (Parthasarathy, 1967, Thm. 2.1).

2.2. Rational belief hierarchies and rational types

Let Θ be a Polish space together with the Borel σ-algebra, $\mathcal{B}_0$. For instance, in a game, each $\theta \in \Theta$ corresponds to a payoff vector (Harsanyi, 1967-68), or a strategy profile (Aumann and Brandenburger, 1995; Tan and Werlang, 1988), or a combination of the two. Throughout the paper, we refer to Θ as the underlying – or fundamental – space of uncertainty. Let $I = \{a, b\}$ be the set of agents, with typical elements i and j.[6]

Each agent forms rational beliefs about Θ (first order rational beliefs), rational beliefs about the opponent's first order rational beliefs (second order rational beliefs), and so on. Such a sequence is called a *rational belief hierarchy*. Formally, consider the sequence

$$
\begin{aligned}
\Theta_0 &:= \Theta \\
\Theta_1 &:= \Theta_0 \times \Delta^{\mathbb{Q}}(\Theta_0) \\
&\vdots \\
\Theta_{k+1} &:= \Theta_k \times \Delta^{\mathbb{Q}}(\Theta_k) \\
&\vdots
\end{aligned}
$$

A *rational belief hierarchy* is a sequence $(\pi_1, \pi_2, \dots)$, with $\pi_k \in \Delta^{\mathbb{Q}}(\Theta_{k-1})$ denoting the k-th order rational beliefs. Let

$$
T_0^Q := \prod_{k=0}^{\infty} \Delta^{\mathbb{Q}}(\Theta_k) \tag{2}
$$

denote the space of all rational belief hierarchies, endowed with the product topology.

In general, belief hierarchies are large and complex objects, and as such it is hard directly working with them. Harsanyi (1967-68) was the first one to circumvent this problem by proposing a compact way of expressing belief hierarchies, known in the literature as the *type space model*. Formally, this model consists of a tuple $(\Theta, T_a, T_b, g_a, g_b)$, where T_i is a Polish space of types with typical element t_i, and $g_i : T_i \to \Delta(\Theta \times T_j)$ is a continuous function. In a type space, each $t_i \in T_i$ is associated with a unique *belief hierarchy*.[7]

We say that a type space model is *terminal* whenever for every belief hierarchy there exists a type in T_i inducing it. Moreover, we call a type space model *complete*, if g_i is surjective, implying that every measure in $\Delta(\Theta \times T_j)$ is the image of some type in T_i. Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Mertens et al. (1994) showed that Harsanyi's framework is sufficiently rich to model all instances of interactive uncertainty, in that there is a type space model $(\Theta, T_a^*, T_b^*, g_a^*, g_b^*)$, with $T_a^* = T_b^* = T^*$ and $g_a^* = g_b^* = g^*$, which is both complete and terminal. This construction is called the *universal* type space

---

[6]Our analysis can be directly generalized to any finite set of agents, in which case we obviously allow for correlated beliefs, as usual.

[7]For a detailed presentation on how the entire belief hierarchy is derived from a type space model, we refer to Siniscalchi (2007).

model.[8]

The first natural question arising at this point is whether we can extend their result to the case of rational belief hierarchies. In other words, is there a universal type space model of rational belief hierarchies in the same line as the standard result of Brandenburger and Dekel (1993)?

As usual, we further restrict rational belief hierarchies so that they satisfy, not only coherency, but also common certainty in coherency. Let T^Q denote the set of rational belief hierarchies satisfying coherency and common certainty in coherency. Henceforth, whenever we write "rational belief hierarchies" or "rational types", we implicitly refer to elements of T^Q, thus omitting to explicitly say that they satisfy coherency and common certainty in coherency. Following Brandenburger and Dekel (1993), Tsakas (2012) proved the existence of a *terminal type space model of rational belief hierarchies*, implying that every rational belief hierarchy is identified by a Borel probability measure on $\Theta \times T^Q$, via the injective function

$$g : T^Q \to \Delta(\Theta \times T^Q). \tag{3}$$

It is rather easy to see that the function g is in fact the same as g^* from Brandenburger and Dekel (1993), but restricted to the domain of rational belief hierarchies. Throughout the paper, we treat g and g^* as the same function. Note that g is natural mapping, in the sense that every rational type is associated with a probability measure over $\Theta \times \prod_{k=0}^{\infty} \Delta^{\mathbb{Q}}(\Theta_k)$ that has the property that its marginal distribution over Θ_{k-1} coincides with the k-th order beliefs induced by this hierarchy, i.e., for every $(\pi_1, \pi_2, \dots) \in T^Q$

$$\text{marg}_{\Theta_{k-1}} g(\pi_1, \pi_2, \dots) = \pi_k. \tag{4}$$

Obviously, this type space model is not complete, as there are Borel probability measures in $\Delta(\Theta \times T^Q)$ which are not the image of any rational type. The latter is not surprising, as one can easily see that there exist probability measures $\pi \in \Delta(\Theta \times T^Q)$ with $\text{marg}_\Theta \pi \notin \Delta^{\mathbb{Q}}(\Theta)$, e.g., a measure with $\pi(\{\theta\} \times T^Q) = \sqrt{2}/2$.

More interestingly, Tsakas (2012) showed that there exist types that are mapped via g to non-rational probability measures over $\Theta \times T^Q$. Throughout the paper, we call the rational types that do not exhibit this property *universally rational*, i.e., a type $t \in T^Q$ is universally rational if and only if $g(t) \in \Delta^{\mathbb{Q}}(\Theta \times T^Q)$.

Proposition 2 (Tsakas, 2012). *There exists some $t \in T^Q$ which is not universally rational.*

In fact, the types $t \in T^Q$ that are not universally rational, attach a non-rational probability to Borel events that are expressible if the agent's language is countably generated.[9] This implies that in the existence of a countable generated language modeled by the Borel σ-algebra, agents may not be able to avoid using non-rational beliefs for some event in their language even if their entire belief hierarchy contains only rational beliefs. In this paper we identify the conditions under which a rational type does not exhibit this property. In other words, we characterize the set of universally rational types.

---

[8]Heifetz (1993) generalized this representation result to cases where the underlying space of uncertainty is Hausdorff and beliefs are modeled by tight probability measures, while Heifetz and Samet (1998) considered a purely measurable underlying space of uncertainty. Finally, Pintér (2010) studied the existence of a universal type space for arbitrary topological spaces.

[9]In fact Tsakas (2012, Section 5.1) discusses the case of a finitely generated language.

3. *N-rational probability measures*

Recall from Proposition 1 that every rational measure has a finite support. Hence, for every rational measure there exists some finite $N \subseteq \mathbb{N}$ such that the probability assigned to every Borel event can be written as fraction with the denominator belonging to N. Formally, let

$$\mathbb{Q}_N := \{ m/n \in \mathbb{Q} : m = 0, \dots, n \; ; \; n \in N \},$$

contain all rational numbers with this property. Then, define the set of all rational probability measures that take values only in $\mathbb{Q}_N$.

Definition 2. We define the set of *N-rational probability measures* by

$$\Delta^N(X) := \left\{ \mu \in \Delta(X) \; : \; \mu(B) \in \mathbb{Q}_N, \forall B \in \mathcal{B} \right\}. \tag{5}$$

Obviously, if $N \subseteq M \subseteq \mathbb{N}$, then $\Delta^N(X) \subseteq \Delta^M(X) \subseteq \Delta^{\mathbb{N}}(X) = \Delta^{\mathbb{Q}}(X)$, implying that *N*-rational probability measures are in fact rational. Recall that a rational probability measure describes the beliefs of an agent who avoids to use complex beliefs, such as the ones that attach a non-rational probability to some Borel event E, even though her language allows her to understand what it means for instance to put probability $\sqrt{2}/2$ to E. Thus, *N*-rational probability measures describe the beliefs of an agent who forms even simpler beliefs, in the sense that she only divides with a finite set of denominators. For instance, consider an agent who expresses her beliefs only in terms of percentages. In this case, the agent's beliefs can be modeled with an *N*-rational measure, where $N = \{100\}$. Similarly, if the agent expresses beliefs in terms of decimals with up to 3 digits, then $N = \{1000\}$.[10]

Proposition 3. $\Delta^N(X)$ *is closed in* $\Delta(X)$.

4. *N-rational belief hierarchies and universally rational belief hierarchies*

In the previous section, we introduced the notion of *N*-rational probability measures. Now, we consider agents who form *N*-rational beliefs for some finite $N \subseteq \mathbb{N}$ at every order of their hierarchy. In this case we say that the belief hierarchy is *N*-rational.

Notice that in principle we do not require the agent to believe that everybody else's beliefs are restricted by the same N, i.e., agent i's belief hierarchy may be *N*-rational and still attach positive probability to the opponent's beliefs being *M*-rational for some $M \supsetneq N$, or even believe that her opponent's beliefs are just rational.

Formally, for some finite $N \subseteq \mathbb{N}$, consider a sequence $(\pi_1, \pi_2, \dots) \in \prod_{k=0}^{\infty} \Delta^N(\Theta_k)$, where each $\pi_k \in \Delta^N(\Theta_{k-1})$ denotes the *N*-rational *k*-th order beliefs. In other words, the agent forms *N*-rational beliefs about Θ (first order *N*-rational beliefs), *N*-rational beliefs about the opponent's first order rational beliefs (second

---

[10]Recent experimental evidence indicates that subjects have a tendency to rely on decimal approximations of non-rational numbers (Sirotic and Zazkis, 2007b).

N-rational order beliefs), and so on. As we have already mentioned above, observe that the second order beliefs may attach positive probability to the opponent holding some rational, but not necessarily N-rational first order beliefs, e.g., though the agent expresses her beliefs in terms of percentages, she may still assign positive probability to her opponent believing some Borel event $E \subseteq \Theta$ with probability 1/3.

Let $T_0^N := \prod_{k=0}^{\infty} \Delta^N(\Theta_k)$ denote the space of *N-rational belief hierarchies*. Then, impose the usual restriction of coherency and common certainty in coherency,

$$T^N := T_0^N \cap T^Q. \tag{6}$$

Types in T^N are called *N-rational*.[11]

The following result provides a type space model of N-rational belief hierarchies. That is, if an agent's rational belief hierarchy is further restricted by some finite $N \subseteq \mathbb{N}$, then the agent attaches an N-rational probability to every Borel event in $\Theta \times T^Q$ conditional on her own type. Moreover, every N-rational probability measure on $\Theta \times T^Q$ is the image of some N-rational type.

Theorem 1. $g : T^N \to \Delta^N(\Theta \times T^Q)$ *is homeomorphic.*

The previous result implies that a rational belief hierarchy is universally rational if and only if it is N-rational for some finite $N \subseteq \mathbb{N}$. To see this, let $\mathcal{U} := \{ N \subseteq \mathbb{N} : 0 < |N| < \infty \}$ denote the collection of all non-empty finite subsets of $\mathbb{N}$,[12] and observe that for every separable and metrizable space X it is the case that $\Delta^Q(X) = \bigcup_{N \in \mathcal{U}} \Delta^N(X)$. Therefore it follows that

$$
\begin{aligned}
\Delta^Q(\Theta \times T^Q) &= \bigcup_{N \in \mathcal{U}} \Delta^N(\Theta \times T^Q) \\
&= \bigcup_{N \in \mathcal{U}} g(T^N).
\end{aligned}
$$

That is, a rational type assigns to every Borel event in $\Theta \times T^Q$ a rational probability if and only if this type is N-rational for some $N \subseteq \mathbb{N}$.

The following example presents a rational belief hierarchy that is not universally rational, and illustrates that this type is not N-rational for any $N \in \mathcal{U}$.

Example 1. This example is taken from the proof of Theorem 1 in Tsakas (2012): Take two arbitrary $\theta_1, \theta_2 \in \Theta$, and consider the following sequence:

$$
\begin{aligned}
P_1 &:= \{ p_1 \in \Delta^Q(\Theta_0) : p_1(\theta) = 1, \text{ for some } \theta \in \{\theta_1, \theta_2\} \} \\
P_2 &:= \{ p_2 \in \Delta^Q(\Theta_1) : p_2(\theta, p_1) = 1, \text{ for some } (\theta, p_1) \in \{\theta_1, \theta_2\} \times P_1 \} \\
&\vdots \\
P_k &:= \{ p_k \in \Delta^Q(\Theta_{k-1}) : p_k(\theta, p_1, \dots, p_{k-1}) = 1, \text{ for some } (\theta, p_1, \dots, p_{k-1}) \in \{\theta_1, \theta_2\} \times P_1 \times \dots \times P_{k-1} \} \\
&\vdots
\end{aligned}
$$

---

[11]Throughout the paper, we use the term N-rational belief hierarchies (N-rational types) to refer to elements of T^N rather than T_0^N, thus implicitly considering only hierarchies that satisfy coherency and common certainty in coherency.

[12]Notice that $\mathcal{U}$ is countable.

Let T_p be the set of types $(p_1, p_2, \dots) \in \prod_{k>0} P_k$ that satisfy coherency and common certainty in coherency. Observe that T_p has the same cardinality as $\{0,1\}^{\mathbb{N}}$, implying that it is uncountable. Now, consider a belief hierarchy $(\pi_1, \pi_2, \dots)$ such that π_k is uniformly distributed over $\Theta \times \text{proj}_{P_1 \times \cdots \times P_{k-1}} T_p$ for every $k > 0$. Notice that $(\pi_1, \pi_2, \dots)$ satisfies coherency and common certainty in coherency. Moreover, by construction $(\pi_1, \pi_2, \dots) \in T^Q$, and therefore $g(\pi_1, \pi_2, \dots) \in \Delta(\Theta \times T^Q)$. However, observe that $g(\pi_1, \pi_2, \dots)$ has an infinite support, and therefore by Proposition 1, $g(\pi_1, \pi_2, \dots) \notin \Delta^{\mathbb{Q}}(\Theta \times T^Q)$, implying that $(\pi_1, \pi_2, \dots)$ is not universally rational.

Now, notice that $(\pi_1, \pi_2, \dots)$ is not N-rational for any finite N, since $\pi_1 \in \Delta^{\{1\}}(\Theta_0)$, $\pi_2 \in \Delta^{\{2\}}(\Theta_1)$, $\pi_3 \in \Delta^{\{4\}}(\Theta_2)$, and so on. In fact, at every order of beliefs the agent uses a denominator that has not been used in lower order beliefs. Hence, there is no finite $N \subseteq \mathbb{N}$ such that $\pi_k \in \Delta^N(\Theta_{k-1})$ for all $k > 0$. ◄

Appendix A. Proofs

Proof of Proposition 3. It suffices to show that an arbitrary convergent sequence $\{\mu_k\}$ of elements of $\Delta^N(X)$ has its limit in $\Delta^N(X)$, i.e., if $\mu_k \xrightarrow{w^*} \mu$, then $\mu \in \Delta^N(X)$. Let $\bar{n} := \max_{n \in N} n$. Let also $d : X \times X \to \mathbb{R}$ be a metric compatible with the topology on X, and for every $x \in X$ and $\delta > 0$, define an open neighborhood of x as $B(x, \delta) := \{x' \in X : d(x, x') < \delta\}$. Consider an arbitrary $x \in X$, and suppose there is some $\delta > 0$ such that there are finitely many $k > 0$ with $\mu_k(B(x, \delta)) > 0$. Then, obviously, there are infinitely many $k > 0$ such that $\mu_k(B(x, \delta)) = 0$, implying that $\liminf \mu_k(B(x, \delta)) = 0$. Hence, it follows from $\mu_k \xrightarrow{w^*} \mu$ that $\mu(B(x, \delta)) \leq \liminf \mu_k(B(x, \delta)) = 0$ (Aliprantis and Border, 1994, Thm. 15.3), implying that $x \notin \text{supp}(\mu)$. If, on the other hand, for every $\delta > 0$ there are infinitely many $k > 0$ such that $\mu_k(B(x, \delta)) > 0$, it follows from $\mu_k \in \Delta^N(X)$ that there are infinitely many $k > 0$ such that $\mu_k(\overline{B(x, \delta)}) \geq 1/\bar{n}$, where $\overline{B(x, \delta)} := \{x' \in X : d(x, x') \leq \delta\}$ is the closure of $B(x, \delta)$. Therefore, $\mu(\overline{B(x, \delta)}) \geq \limsup \mu_k(\overline{B(x, \delta)}) \geq 1/\bar{n}$ (Aliprantis and Border, 1994, Thm. 15.3). Now, consider a sequence of positive reals $\{\delta_n\}$ with $\delta_n \downarrow 0$, which induces a sequence of Borel events $\{\overline{B(x, \delta_n)}\}$ such that $\limsup_{n>0} \overline{B(x, \delta_n)} = \{x\}$. Then, it follows from $\mu(\limsup_{n>0} \overline{B(x, \delta_n)}) \geq \limsup_{n>0} \mu(\overline{B(x, \delta_n)}) \geq 1/\bar{N}$ (Billingsley, 1995, Thm. 4.1) that $\mu(\{x\}) \geq 1/\bar{n}$. Hence, $x \in \text{supp}(\mu)$ if and only if $\mu(\{x\}) \geq 1/\bar{n}$, implying that $\text{supp}(\mu)$ is finite. Let $x \in \text{supp}(\mu)$. It follows from Aliprantis and Border (1994, Thm. 15.3) that for every $\delta > 0$,

$$
\begin{aligned}
\mu(\overline{B(x, \delta)}) &\geq \limsup \mu_k(\overline{B(x, \delta)}) \\
&\geq \limsup \mu_k(B(x, \delta)) \\
&\geq \liminf \mu_k(B(x, \delta)) \\
&\geq \mu(B(x, \delta)). \tag{A.1}
\end{aligned}
$$

Since $\text{supp}(\mu)$ is finite, there is some $\rho > 0$ such that $x' \notin B(x, \rho)$ for any $x' \in \text{supp}(\mu) \setminus \{x\}$, implying that $\mu(\overline{B(x, \delta)}) = \mu(B(x, \delta)) = \mu(\{x\})$ for every $\delta < \rho$. Hence, it follows from (A.1) that $\mu(\{x\}) = \lim \mu_k(B(x, \delta))$. Finally, since the sequence $\{\mu_k(B(x, \delta))\}$ contains only elements of the finite set $\mathbb{Q}_N$, it follows that $\lim \mu_k(B(x, \delta)) \in \mathbb{Q}_N$. □

Proof of Theorem 1. Showing that $\Delta^N(\Theta \times T^Q) \subseteq g(T^N)$ follows directly by construction. Then, we show that $g(T^N) \subseteq \Delta^N(\Theta \times T^Q)$. Consider an arbitrary $(\pi_1, \pi_2, \dots) \in T^N$. Since $T^N \subseteq T^Q$, it follows from Kolmogorov Extension Theorem that there is some $\pi \in \Delta(\Theta \times T^Q)$ such that $g(\pi_1, \pi_2, \dots) = \pi$. It suffices to show that $\pi \in \Delta^N(\Theta \times T^Q)$. For each $k \geq 0$, let $\mathcal{B}_k$ denote the Borel σ-algebra in Θ_k. Since π extends every π_{k+1}, it follows that for every $B_k \in \mathcal{B}_k$,

$$\pi_{k+1}(B_k) = \pi\Big(B_k \times \prod_{\ell=k}^{\infty} \Delta^{\mathbb{Q}}(\Theta_\ell)\Big). \tag{A.2}$$

Observe that every Borel event $B \subseteq \Theta \times T^Q$ is also Borel in $\Theta \times \prod_{k=0}^{\infty} \Delta^{\mathbb{Q}}(\Theta_k)$, and

$$B = \bigcap_{k=0}^{\infty}\Big(\mathrm{proj}_{\Theta_k} B \times \prod_{\ell=k}^{\infty} \Delta^{\mathbb{Q}}(\Theta_\ell)\Big). \tag{A.3}$$

Then, it follows from Billingsley (1995, Thm. 4.1), together with Eq. (A.3), that

$$
\begin{aligned}
\pi(B) \quad &= \quad \lim_{k \to \infty} \pi\Big(\mathrm{proj}_{\Theta_k} B \times \prod_{\ell=k}^{\infty} \Delta^{\mathbb{Q}}(\Theta_\ell)\Big) \\
&\overset{(A.2)}{=} \quad \lim_{k \to \infty} \pi_{k+1}(\mathrm{proj}_{\Theta_k} B).
\end{aligned}
\tag{A.4}
$$

Since $(\mathrm{proj}_{\Theta_k} B) \in \mathcal{B}_k$ and $\pi_{k+1} \in \Delta^N(\Theta_k)$, it follows that $\pi_{k+1}(\mathrm{proj}_{\Theta_k} B) \in \mathbb{Q}_N$. Since $\{\pi_{k+1}(\mathrm{proj}_{\Theta_k} B)\}_{k>0}$ is a convergent sequence taking finitely many values, it follows that the limit converges to one of these values. Therefore, $\pi(B) \in \mathbb{Q}_N$, which proves that $\pi \in \Delta^N(\Theta \times T^Q)$. Thus, $g : T^N \to \Delta^N(\Theta \times T^Q)$ is bijective.

In order to prove that g is continuous, consider a sequence $\{(\pi_1^n, \pi_2^n, \dots)\}_{n>0}$ of elements of T^N that weakly converges to $(\pi_1, \pi_2, \dots)$ which by Proposition 3 also belongs to T^N, and let $\pi^n := g(\pi_1^n, \pi_2^n, \dots)$ and $\pi := g(\pi_1, \pi_2, \dots)$. Then, it suffices to show that the sequence $\{\pi^n\}_{n>0}$ weakly converges to π. Consider an arbitrary closed set $B \subseteq \Theta \times T^Q$, and notice that $B_k := \mathrm{proj}_{\Theta_k} B$ is closed in Θ_k. Hence, it follows from the Portmanteau Theorem (e.g., see Aliprantis and Border, 1994, Thm. 15.3) that

$$\limsup_{n>0} \pi_{k+1}^n(B_k) \leq \pi_{k+1}(B_k) \tag{A.5}$$

for all $k \geq 0$. Observe that for every $k \geq 0$, it is the case that $B \subseteq B_k \times \prod_{\ell=k}^{\infty} \Delta^{\mathbb{Q}}(\Theta_\ell)$, and therefore we obtain that for every $k \geq 0$,

$$
\begin{aligned}
\limsup_{n>0} \pi^n(B) \quad &\leq \quad \limsup_{n>0} \pi^n\Big(B_k \times \prod_{\ell=k}^{\infty} \Delta^{\mathbb{Q}}(\Theta_\ell)\Big) \\
&= \quad \limsup_{n>0} \pi_{k+1}^n(B_k) \\
&\overset{(A.5)}{\leq} \quad \pi_{k+1}(B_k).
\end{aligned}
$$

Since the latter is true for every $k \geq 0$, it follows that

$$
\begin{aligned}
\limsup_{n>0} \pi^n(B) \quad &\leq \quad \lim_{k \to \infty} \pi_{k+1}(B_k) \\
&\overset{(A.4)}{=} \quad \pi(B).
\end{aligned}
$$

Therefore, it follows again from the Portmanteau Theorem that $\pi^n \xrightarrow{w^*} \pi$.

Proving that g^{-1} is continuous is straightforward: Consider a sequence $\{\pi^n\}_{n>0}$ of elements of $\Delta^N(\Theta \times T^Q)$ such that $\pi^n \xrightarrow{w^*} \pi$, and notice that $\mathrm{marg}_{\Theta_k} \pi^n \xrightarrow{w^*} \mathrm{marg}_{\Theta_k} \pi$ for all $k \geq 0$. Hence, it follows that $g^{-1}(\pi^n) \xrightarrow{w^*} g^{-1}(\pi)$ which completes the proof. $\qquad\square$

References

ALIPRANTIS, C. & BORDER, K. (1994). *Infinite dimensional analysis*. Springer Verlag, Berlin.

AUMANN, R.J. (1987). Correlated equilibrium as an expression of Bayesian rationality. *Econometrica* 55, 1–18.

——— (1999). Interactive epistemology II: Probability. *International Journal of Game Theory* 28, 301–314.

AUMANN, R.J. & BRANDENBURGER, A. (1995). Epistemic conditions for Nash equilibrium. *Econometrica* 63, 1161–1180.

BILLINGSLEY, P. (1968). *Convergence of probability measures*. John Wiley & Sons, New York.

——— (1995). *Probability and measure*. John Wiley & Sons, New York.

BRANDENBURGER, A. (2008). Epistemic game theory: complete information. *The New Palgrave Dictionary of Economics*, edited by L. Blume & S. Durlauf, McMillan.

BRANDENBURGER, A. & DEKEL, E. (1987). Rationalizability and correlated equilibria. *Econometrica* 55, 1391–1402.

——— (1993). Hierarchies of beliefs and common knowledge. *Journal of Economic Theory* 59, 189–198.

FAGIN, R. & HALPERN, J. (1994). Reasoning about knowledge and probability. *Journal of the Association for Computing Machinery* 41, 340–367.

GUPTA, A. (2000). *Real and abstract analysis*. Academic Publishers, Calcutta.

HARSANYI, J. (1967-68). Games with incomplete information played by Bayesian players, I-III. *Management Science* 14, 159–182, 320–334, 486–502.

HEIFETZ, A. (1993). The Bayesian formulation of incomplete information — The non-compact case. *International Journal of Game Theory* 21, 329–338

HEIFETZ, A. & MONGIN, P. (2001). Probability logic for type spaces. *Games and Economic Behavior* 35, 31–53.

HEIFETZ, A. & SAMET (1998). Topology-free typology of beliefs. *Journal of Economic Theory* 82, 324–341.

KECHRIS, A. (1995). *Classical descriptive set theory*. Springer Verlag, Berlin.

MERTENS, J.F. & ZAMIR, S. (1985). Formulation of Bayesian analysis for games with incomplete information. *International Journal of Game Theory* 14, 1–29.

MERTENS, J.F., SORIN, S. & ZAMIR, S. (1994). Repeated games: Part A. *CORE Discussion Paper*.

PARTHASARATHY, K.R. (1967). *Probability measures on metric spaces*. AMS Chelsea Publishing, Providence, Rhode Island.

PEREA, A. (2012). *Epistemic game theory: Reasoning and choice*. Cambridge University Press. UK.

PINTÉR, M. (2010). The non-existence of a universal topological type space. *Journal of Mathematical Economics* 46, 223–229.

SINISCALCHI, M. (2007). Epistemic game theory: Beliefs and types. *The New Palgrave Dictionary of Economics*, edited by L. Blume & S. Durlauf, McMillan, forthcoming.

SIROTIC, N. & ZAZKIS, R. (2007a). Irrational numbers: The gap between formal and intuitive knowledge. *Educational Studies in Mathematics* 65, 49–76.

——— (2007b). Irrational numbers on the line – where are they? *International Journal of Mathematical Education in Science and Technology* 38, 477–488.

SRIVASTAVA, S.M. (1991). *A course on Borel sets*. Springer Verlag, Berlin.

TAN, T. & WERLANG, S. (1988). The Bayesian foundations of solution concepts of games. *Journal of Economic Theory* 45, 370–391.

TSAKAS, E. (2012). Rational belief hierarchies. *METEOR Research Memorandum, Maastricht University*.

ZHOU, C. (2010). Probability logic of finitely additive beliefs. *Journal of Logic, Language and Informatics* 19, 247–282.