Theory of Individual and Strategic Decisions

Course manual

Course manual EBC4197
Academic year 2019 – 2020
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Department of Economics
Maastricht University

Contents

1 Introduction

1.1 Aim of the course

The aim of the course is to provide an overview of the standard analytical tools that are used for modelling decision making under uncertainty. Decision Theory focuses on individual decision problems, and the general purpose is twofold. On the one hand, it provides us with a framework within which common choice patterns and individual characteristics can be identified and studied. For instance, it allows us to formally define and measure risk attitudes. On the other hand, it provides us with a natural benchmark to later on study Game Theory, which in turn focuses on decisions in environments with strategic interactions among several agents. The course mainly focuses on rational agents with unlimited reasoning abilities. These agents are used as a benchmark in most economic models. However, we will also briefly refer to empirical violations of this paradigm, thus providing a link to experimental economics and psychology.

1.2 Prerequisites

It is expected that the students who take this course have a minimum level of familiarity with undergraduate mathematics and statistics. As an indication, students are expected to be familiar with the concepts discussed in Chapter 19 (Mathematical Appendix) of *Game Theory: An Introduction* by Steven Tadelis, and in the online Appendix A of *Rational Choice* by Itzhak Gilboa. Although no prior knowledge in microeconomic theory or game theory is necessary, as the course is self-contained, it is recommended (especially for students with no background in economics) to read Chapters 1–3 of *Economics* by Daron Acemoglu, David Laibson and John List, in order to get an idea of the main purposes and the methological underpinnings of economics research.

1.3 Expectations

Though the course is demanding, it is structured in a way such that a student can be successful by regularly attending the lectures and tutorial meetings, and studying according to the schedule described below. A rough estimate of the *self-study time* needed for the course is approximately 20 hours per week. In order to pass the course, the students will need to complete certain tasks (final exam, participation, problem sets, self-evaluation). Some of these tasks will be undertaken in working groups that will be formed at the beginning of the first lecture. There will be 4 working groups per tutorial group. For particularly ambitious students, there will be additional suggested material that will go beyond the basic concept that will be discussed in class, e.g., advanced video lectures, extra problems, additional literature.

2 Course structure

2.1 Description

The course consists of 14 meetings (7 lectures and 7 tutorials).

- **Lectures:** During each lecture a new topic will be discussed. The lecturer will provide an overview of the corresponding theory. Participation in the lectures is not formally mandatory, but nevertheless strongly advised.

- **Tutorial meetings:** There are two types of tutorials, theory sessions and exercise sessions. The aim of the meetings is to deeply understand the topics introduced in the lectures. Participation in the tutorial meetings is compulsory and it is graded.
  - **Theory sessions** are scheduled immediately after the corresponding lectures. The students are expected to have studied the lecture notes (see Appendices) and the designated literature, and to have prepared answers to the questions that appear throughout the notes. At the beginning of each meeting, the tutor will assign some of the questions to each student, and the student will be responsible for providing an answer. These answers will contribute to the participation grade. After the student has answered the question, other students can provide comments and/or additional input to the answer.
  - **Exercise sessions** are scheduled immediately after the theory tutorials. The students are expected to have worked on the designated set of problems before each exercise tutorial. The answers should be typewritten and sent by midnight before the session to ebc4197@gmail.com. Solutions will be submitted in groups that will have been formed by the course coordinator at the beginning of the course. During each exercise tutorial, students will be invited by the tutor to go to the board and present their solution. This will also be part of the participation grade.

2.2 Schedule

The precise schedule of the course is as follows:

1. **Lecture on Decision Theory I** (Elias Tsakas)
2. **Lecture on Decision Theory II** (Elias Tsakas)
3. **Lecture on Decision Theory III** (Elias Tsakas)
4. **Theory Session on Preferences and Utility** (Eveline Vandewal)
   Literature: Appendix B; Bonanno (Decision Making; Ch. 1,2,3)
5. **Theory Session on Choice under uncertainty** (Eveline Vandewal)
   Literature: Appendix C; Bonanno (Decision Making; Ch. 5)
6. **Exercise Session on Decision Theory** (Eveline Vandewal)
   Problems: B.1, B.2, C.1, C.2
7. **Lecture on Strategic Form Games I** (Elias Tsakas)
8. **Lecture on Strategic Form Games II** (Elias Tsakas)
9. **Theory Session on Strategic Form Games** (Eveline Vandewal)
   Literature: Appendix D; Bonanno (Game Theory; Ch. 1,2,6)
10. **Exercise Session on Strategic Form Games** (Eveline Vandewal)
    Problems: D.1, D.2, D.3, D.4
11. **Lecture on Extensive Form Games I** (Elias Tsakas)

12. **Lecture on Extensive Form Games II** (Elias Tsakas)

13. **Theory Session on Extensive Form Games** (Eveline Vandewal)

    Literature: Appendix E; Bonanno (Game Theory; Ch. 3,4)

14. **Exercise Session on Extensive Form Games** (Eveline Vandewal)

    Problems: E.1, E.2, E.3, E.4

2.3 Literature

Throughout the course we will be using various textbooks. For the theory sessions the students are expected to have read the corresponding chapters.

- Main textbooks (used throughout the course):
  - Bonanno, G. (2015). *Game Theory*. Open access textbook (free download).
  - Bonanno, G. (2018). *Decision Making*. Open access textbook (free download).
- Additional textbooks (recommended for further reading):
  - Gilboa, I. (2009). *Theory of Decision under Uncertainty.* Econometric Society Monographs.
  - Rubinstein, A. (2012). *Lecture Notes in Microeconomic Theory: The Economic Agent*. Princeton University Press (free download).
  - Tadelis, S. (2013). *Game Theory: An Introduction.* Princeton University Press.

3 Performance assessment

The final grade will be calculated based on the performance in the following tasks (with the corresponding weights in parenthesis):

- **Final exam (75%):** The final exam consists of both theoretical questions and problems of gradually increasing difficulty.
- **Problem sets (15%):** There are 3 problem sets, one for each exercise session. Your answers to the problem sets should be typewritten in a clear and professional way. Handwritten answers will not be accepted. The answers should be sent in pdf format to ebc4197@gmail.com by midnight before the corresponding exercise tutorial. The deadline is strict and late submissions will not be accepted. The answers should be submitted in working groups. Individual answers will not be accepted. If a member of the group has zero contribution to solving a problem set, it should be reported to the course coordinator.
- **Self-evaluation (3%):** After each exercise session, the answers of the corresponding problems will be published on eleum. Using these answers, together with the answers discussed in class with the tutor, you must go back to your submitted answers (i.e., the ones you sent by midnight the day before) and grade your own problem set on a scale $0-10$. Your self-evaluation will be compared to the grade that your tutor gives you (which will also be on a scale $0-10$), and

if the two are at most 1 point apart from each other then you will receive a bonus of 1% of the full score to your final grade. Note that in order to be eligible for the bonus, your actual grade (i.e., the one you receive by your tutor) must be at least 5/10. Self-evaluation is done individually, and not in groups although you have solved the problems together. It is always possible that you are asked to justify your self-evaluation, by means of explaining why you graded your problem set the way you did. Failure to do so, may result in loosing your bonus.

- **Participation (7%):** In each tutorial meeting you will be assigned some question or problem to answer. You are expected to be able to answer such questions. You are also expected to participate when discussing other questions. Finally, you must be present in at least 5 of the 7 tutorials. If you miss more than 2 tutorials, you fail the course. In extenuating circumstances, you may be allowed to compensate more than 2 absences with some additional assignment. Whether this will be the case, and which the extra assignment will be, is entirely at the discretion of the course coordinator.

The final grade will be on a scale $1 - 10$, rounded to the nearest half point. To pass the course one needs at least 55 points. That is, 53 points and 54 points are downgraded to 5 and therefore do not suffice to pass the course. For the students who fail the course, there will be a resit exam organized by the school. The students who take the resit carry with them the grades received for the other three task. Note that partial grades obtained in this course remain valid for a period of three years, i.e., the academic year in which the grades were obtained plus two subsequent academic years. After the final exam the course coordinator will announce an inspection date during which students can see their exam. Inspection outside the designated times cannot be guaranteed. The same applies to the resit.

4 Contact information

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A Mathematical preliminaries

A.1 Set theory

We start with a grand set X. For a typical element of X is denoted by x. A subset of X is a collection of some (perhaps all) of the elements of X. Whenever an element x belongs to a subset A, we write $x \in A$. Otherwise, we write $x \notin A$. The empty set is denoted by $\emptyset$ and it does not contain any element.

Inclusion: We say that A is a subset of B, and we write $A \subseteq B$, whenever $x \in A$ implies $x \in B$. Obviously, $\emptyset \subseteq A \subseteq X$ for every subset A.

Union: $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

Intersection: $x \in A \cap B$ if and only if $x \in A$ and $x \in B$. We say that A and B are disjoint whenever $A \cap B = \emptyset$.

Complement: A and B are complements whenever $A \cup B = X$ and $A \cap B = \emptyset$.

Set difference: $x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$.

A.2 Calculus

A function $f : X \to Y$ is a mapping that takes each element $x \in X$ and associates it with an element $f(x) \in Y$.

Strictly increasing function: a function $f : \mathbb{R} \to \mathbb{R}$ such that $x > y$ implies $f(x) > f(y)$.

Linear function: a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \alpha + \beta x$ for $\alpha, \beta \in \mathbb{R}$. Whenever $\beta > 0$, it is obviously strictly increasing.

Transformation: take a function $u : \mathbb{R} \to \mathbb{R}$, and another function $f : \mathbb{R} \to \mathbb{R}$. Then, the function $f(u)$ is a transformation of u, and it assigns to each $x \in \mathbb{R}$ value $f(u(x))$.

A.3 Probability theory

We start with a (finite) state space, which is a set X. Each element $x \in X$ is called a state, and it is interpreted as a full description of everything that is relevant, i.e., once we know which state is the true one, all relevant uncertainty has been resolved. A subset $A \subseteq X$ is called an event.

Random variable: it is a real-valued function $u : X \to \mathbb{R}$ that specifies a value to an object of interest at each state.

Probability distribution: it assigns to each $x \in X$ a probability $\mu(x) \in [0,1]$ in a way such that $\sum_{x \in X} \mu(x) = 1$. The probability of an event $A \subseteq X$ is then equal to $\sum_{x \in A} \mu(x)$. The probability of the empty set is 0.

Expected value: for a random variable u and probability distribution μ, the expected value is

$$\mathbb{E}_\mu(u) = \sum_{x \in X} \mu(x) u(x).$$

B Preferences and Utility

Choice domain: we start with an arbitrary set $X \subseteq \mathbb{R}^n$ of (sure) outcomes. Outcomes can be thought as monetary outcomes or as consumption bundles.

(Weak) preference: a binary relation $\succeq \subseteq X \times X$ over pairs of outcomes. We write $x \succeq y$ whenever $(x, y) \in \succeq$, and the interpretation is that the individual finds x to be at least as good as y.

Strict preference: a binary relation $\succ \subseteq X \times X$ defined by $x \succ y$ if and only if $x \succeq y$ and $y \not\succeq x$. The interpretation is that the individual finds x to be strictly better than y.

Indifference: a binary relation $\sim \subseteq X \times X$ defined by $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$.

Question B.1. *Explain the following relations: (a)* $\succ \subseteq \succeq$, *(b)* $\sim \subseteq \succeq$, *(c)* $\succ \cup \sim = \succeq$.

(Weak) upper-contour set of x: $U_{\succeq}(x) := \{y \in X : y \succeq x\}$

(Weak) lower-contour set of x: $L_{\succeq}(x) := \{y \in X : x \succeq y\}$

Indifference curve of x: $I(x) := \{y \in X : x \sim y\}$.

Question B.2. *Provide intuition for* $U_{\succeq}(x)$, $L_{\succeq}(x)$ *and* $I(x)$.

Question B.3. *Is the following statement well-defined:* $U_{\succeq}(x) \subseteq \succeq$*? Explain.*

Question B.4. *Prove that* $I(x) = U_{\succeq}(x) \cap L_{\succeq}(x)$.

Axioms: they are natural assumptions we impose on the preferences of the individual. They provide structure (i.e., internal consistency) to the preferences, which is needed in order to be able to build a systematic theory and form testable hypotheses. From now onwards we assume the following axioms:

(A_1) Completeness: for all $x, y \in X$, it is the case that $x \succeq y$ or $y \succeq x$.

(A_2) Transitivity: for all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.

Question B.5. *Are the following true or false: (a)* $x \in U_{\succeq}(x)$, *(b)* $x \in I(x)$*? Explain.*

Question B.6. *What do the following conditions imply: (a)* $U_{\succeq}(x) \subseteq U_{\succeq}(y)$, *(b)* $U_{\succeq}(x) \subset U_{\succeq}(y)$, *(c)* $U_{\succeq}(x) \cap L_{\succeq}(y) = \emptyset$, *(d)* $U_{\succeq}(x) \cap L_{\succeq}(y) \neq \emptyset$*? Explain.*

Problem B.1. *Prove that a preference relation* $\succeq$ *over* X *satisfies completeness and transitivity if and only if the corresponding strict preference relation* $\succ$ *satisfies*

(A'_1) *Asymmetry: For any two outcomes* $x, y \in X$ *it cannot be the case that* $x \succ y$ *and* $y \succ x$ *hold simultaneously.*

(A'_2) *Negative transitivity: For every three outcomes* $x, y, z \in X$ *with* $x \succ y$*, it is the case that* $x \succ z$ *or* $z \succ y$.

Question B.7 (Challenging). *Prove the following statement and provide some intuition: if* X *is a finite set and* $\succeq$ *satisfies completeness and transitivity, then there exists a most-preferred and a least-preferred outcome (not necessarily unique).*

Utility representation of $\succeq$: a function $u : X \to \mathbb{R}$ such that, for all $x, y \in X$

$$x \succeq y \Leftrightarrow u(x) \geq u(y).$$

Intuitively a utility representation is an (unambiguous) translation from the language of the preferences to the language of mathematics.

Question B.8. *Provide intuition for the following statement: if $u : X \to \mathbb{R}$ is a utility representation of $\succeq$ and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then the function $v : X \to \mathbb{R}$ which is defined by $v(x) = f(u(x))$ is also a utility representation of $\succeq$. Explain why it holds.*

Question B.9 (Challenging)**.** *Prove the following statement and provide some intuition: if X is a finite set and $\succeq$ satisfies completeness and transitivity, then there exists a utility representation.*

Problem B.2. *Let $\succeq$ be a complete and transitive preference relation on X. Then, prove that $u : X \to \mathbb{R}$ is a utility representation of $\succeq$ if and only if the following two conditions hold:*

(i) For all $x, y \in X$ with $x \succ y$, it is the case that $u(x) > u(y)$.

(ii) For all $x, y \in X$ with $x \sim y$, it is the case that $u(x) = u(y)$.

We now consider some additional axioms:

(A_3) Continuity: for all $x \in X$, it is the case that $U_{\succeq}(x)$ and $L_{\succeq}(x)$ are closed sets in $\mathbb{R}^n$.

Question B.10. *Provide intuition for (A_3).*

(A_0) Monotonicity: for all $x = (x_1, \dots, x_n) \in X$ and $x = (y_1, \dots, y_n) \in X$, if $x_k \geq y_k$ for all $k = 1, \dots, n$ then $x \succeq y$.

Question B.11. *Provide intuition for (A_0).*

Theorem B.1. *A preference relation over a convex set $X \subseteq \mathbb{R}^n$ satisfies (A_1) completeness, (A_2) transitivity and (A_3) continuity, if and only if, it has a continuous utility representation. Moreover, if it satisfies (A_0) monotonicity, then the utility function is increasing (in every dimension).*

Question B.12. *Take the functions $u(x) = \ln(1 + x)$ and $v(x) = x^2$. Verify that both are representations of the same preferences over $\mathbb{R}_+$, and that these preferences satisfy (A_0) − (A_3). Relate your answer to the one of Question B.8.*

Example B.1. (Lexicographic preferences). Take $X = \mathbb{R}^2_+$ to be the set of all (non-negative) consumption bundles of two goods. Let the preference relation $\succeq$ be defined as follows: For each pair of alternatives $x = (x_1, x_2)$ and $y = (y_1, y_2)$ the agent first compares the first coordinate and prefers the alternative with the highest value, completely disregarding the second coordinates. The agent will use the second coordinates only to break the ties resulted when comparing the first coordinates. In other words, the agent cares infinitely more about the first coordinate than the second one. Formally,

$$x \succeq y \Leftrightarrow \text{ either } x_1 > y_1 \text{ or simultaneously } x_1 = y_1 \text{ and } x_2 \geq y_2. \tag{B.1}$$

We call these preferences lexicographic. ◁

Question B.13. *Verify that the lexicographic preferences of the previous example satisfy completeness, transitivity and monotonicity.*

Question B.14 (Really challenging). *Explain the following statement (by means of an intuitive argument): The lexicographic preferences of the previous example do not have a utility representation. Hint: it is not possible to fit uncountably many disjoint intervals in* $\mathbb{R}$. *Formally, suppose that to every number* $z \in \mathbb{R}_+$ *you associate some interval* $[a_z, b_z]$. *Then, it will necessarily be the case that you can find* $z, z' \in \mathbb{R}_+$ *such that* $[a_z, b_z] \cap [a_{z'}, b_{z'}] \neq \emptyset$.

Question B.15 (Very challenging). *Explain the following statement (graphically): The lexicographic preferences of the previous example do not satisfy continuity.*

Decision problem: a tuple $((X, \succeq), A)$, where $A \subseteq X$ is a set of available choices (sometimes called actions). Whenever it is obvious which is our $(X, \succeq)$, the decision problem is identified by A.

Rational choice/Optimal choice in A: a choice $x \in A$ such that $x \succeq y$ for all $y \in A$.

Question B.16 (Challenging). *Let* $A := \{(x_1, x_2) \in \mathbb{R}^2_+ : p_1x_1 + p_2x_2 \leq c\}$ *be the consumption bundles that satisfy the consumer's budget constraint (where* $c > 0$ *is the consumer's budget). Explain. If the consumer's preferences over* $X = \mathbb{R}^2$ *satisfy* $(A_1) - (A_3)$, *is there a rational choice? What is the implication for the rational choice if* (A_0) *monotonicity is also assumed?*

C Choice under uncertainty

General aim: understand how people choose under uncertainty. Interesting questions of economic relevance that we want to answer are for instance: "how can we identify an individual's risk attitudes" or "how do we elicit an individual's (latent) subjective beliefs"?

Choice domain: we use two different domains of alternatives, depending on the type of uncertainty the individual faces. In particular, we use the set of lotteries to study decisions under objective uncertainty (risk) and the set of acts to study decisions under subjective uncertainty. In both cases, the set of outcomes is assumed to be monetary, i.e., $X \subseteq \mathbb{R}$.

Question C.1. *What is the difference between objective uncertainty and subjective uncertainty?*

Lottery: a random experiment p that yields each monetary outcome $x \in X \subseteq \mathbb{R}$ with some known probability $p(x)$. Even when X is infinite, only finitely many outcomes receive positive probability. So, a lottery p is identified by the probabilities it assigns to each outcome, and it is typically denoted by $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K)$ where $x_1, \ldots, x_K \in X$ and $p(x_1) + \cdots + p(x_K) = 1$. The set of all lotteries is denoted by $\mathcal{L}(X)$, and this will be our set of alternatives. Whenever the set of outcomes (over which the lotteries are defined) is clear from the context, we omit X and we simply write $\mathcal{L}$.

Question C.2. *Explain the following statement: A lottery is a probability distribution over* $\mathbb{R}$*. In this sense, only the underlying probabilities matter and not the actual random experiment. Provide an example of two different random experiments that induce the same lottery.*

Preferences over lotteries: a binary relation $\succeq \subseteq \mathcal{L} \times \mathcal{L}$. Strict preferences and indifference are defined as above.

Question C.3. *Explain the following statement: Each outcome $x \in X$ is a degenerate lottery.*

Expected Utility: for a utility function $u : X \to \mathbb{R}$ over outcomes, the expected utility of a lottery $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K) \in \mathcal{L}$ is equal to

$$\mathbb{E}_p(u) = \sum_{k=1}^{K} p(x_k) u(x_k).$$

von Neumann-Morgenstern Expected Utility (vNM EU): a utility function $u : X \to \mathbb{R}$ is a vNM EU representation of preferences $\succeq$ over lotteries if, for every pair of lotteries $p, q \in \mathcal{L}$,

$$p \succeq q \Leftrightarrow \mathbb{E}_p(u) \geq \mathbb{E}_q(u).$$

vNM axioms: There are axioms that guarantee the existence of a vNM EU representation (viz., $(A_1) - (A_3)$ that we have already discussed, plus one new axiom, called (A_4) independence). If additionally, the preferences satisfy monotonicity, then u is increasing. We will not discuss these axioms in further detail.

Question C.4. *Assume that the preferences $\succeq$ over $\mathcal{L}$ are represented by a strictly increasing vNM EU function. Then, prove formally and provide intuition for the following statement: If $x_1 > x_2$ and $\alpha > \beta$, then $\big(\alpha \times x_1, (1-\alpha) \times x_2\big) \succ \big(\beta \times x_1, (1-\beta) \times x_2\big)$.*

Problem C.1. *Assume that the preferences $\succeq$ over $\mathcal{L}$ are represented by a strictly increasing vNM EU function, and consider five monetary payoffs $x_1 > x_2 > x_3 > x_4 > x_5$. Prove formally and provide intuition for the following statements:*

(a) There exist real numbers $\alpha \in (0,1)$ and $\beta \in (0,1)$ such that

$$\big(\alpha \times x_1, (1-\alpha) \times x_5\big) \sim \big(\beta \times x_2, (1-\beta) \times x_4\big).$$

(b) For any two real numbers $\alpha \in (0,1)$ and $\beta \in (0,1)$ it is the case that

$$\big(\alpha \times x_1, (1-\alpha) \times x_2\big) \succeq \big(\beta \times x_3, (1-\beta) \times x_2\big).$$

Question C.5. *Provide a formal argument and an intuition for the following statement: Assume that $u : X \to \mathbb{R}$ is a vNM EU representation. Then, $v : X \to \mathbb{R}$ is also a vNM EU representation if and only if $v(x) = \alpha + \beta u(x)$ for all $x \in X$, where $\alpha \in \mathbb{R}$ and $\beta > 0$. Relate your answer to the ones of Questions B.8 and B.12.*

Risk Attitudes: An individual is risk-averse if the vNM EU function u is concave. He is risk-seeking if u is convex. He is risk-neutral if u is linear.

Question C.6. *Provide intuition for the previous classification of risk-attitudes.*

Question C.7. *Explain the following statement: Risk attitudes remain the same if we take a linear strictly increasing transformation $v(x) = \alpha + \beta u(x)$ (like in Question B.12). However, this is not necessarily true if we take an arbitrary strictly increasing transformation $v(x) = f(u(x))$ (like in Question B.8), although both u and v represent the same preferences.*

Question C.8. *What is the difference between ordinal and cardinal utility? Relate your answer to the one of Question C.7.*

Set of states: (finitely many) contingencies that the individual is uncertain about. There are no objective probabilities describing this uncertainty. Each state is denoted by ω and the set of all states is denoted by Ω.

Act: it is a contract $f : \Omega \to \mathcal{L}$ that pays one lottery at each state. Thus, $f(\omega)$ is the lottery that the individual will receive (according to the act f) if the state ω is realized. Note that $f(\omega)$ can in principle be a degenerate lottery (i.e., an outcome $x \in X$). Hence, the act is a two-stage procedure: first the state is realized and then an outcome from the corresponding lottery is drawn. The set of all acts is denoted by $\mathcal{F}$.

Preferences over acts: a binary relation $\succeq \subseteq \mathcal{F} \times \mathcal{F}$. Strict preferences and indifference are defined as above.

Question C.9. *Explain the following statement: Each lottery $p \in \mathcal{L}$ is a constant act that does not depend on the state.*

Question C.10. *Explain the following statement: For a given vNM utility function u and an act f, we view $u(f) : \Omega \to \mathbb{R}$ as a random variable that associates each state ω with the vNM expected utility $\mathbb{E}_{f(\omega)}(u)$.*

Subjective Beliefs: a subjective belief is a probability distribution μ over Ω, with $\mu(\omega)$ denoting the probability that the individual attaches to the state $\omega \in \Omega$.

Subjective Expected Utility: for a vNM utility function $u : X \to \mathbb{R}$ and a belief μ over the set of states Ω, the expected utility of an act $f \in \mathcal{F}$ is equal to

$$\mathbb{E}_\mu(u(f)) = \sum_{\omega \in \Omega} \mu(\omega)\mathbb{E}_{f(\omega)}(u).$$

Anscombe-Aumann Subjective Expected Utility (SEU): a vNM utility function $u : X \to \mathbb{R}$ together with a subjective belief μ constitute a SEU representation of preferences $\succeq$ over acts if, for every pair of acts $f, g \in \mathcal{F}$,

$$f \succeq g \Leftrightarrow \mathbb{E}_\mu(u(f)) \geq \mathbb{E}_\mu(u(g)).$$

AA axioms: There are axioms that guarantee the existence of a SEU representation (viz., $(A_1)-(A_4)$ that we have already discussed, plus two new axioms, called (A_5) AA monotonicity and (A_6) non-triviality). If additionally, the preferences satisfy (A_0) monotonicity, then u is increasing. We will only elaborate a bit on (A_5).

Dominance: we say that f strictly dominates g whenever $f(\omega) \succ g(\omega)$ for every $\omega \in \Omega$. We say that f weakly dominates g whenever $f(\omega) \succeq g(\omega)$ for every $\omega \in \Omega$ with the preference being strict in at least one state.

Question C.11. *Provide intuition for strict and weak dominance.*

(A_5) AA monotonicity: If f weakly dominates g, then $f \succeq g$.

(A_5') Strict AA monotonicity: If f strictly dominates g, then $f \succ g$.

Question C.12. *Provide intuition for (A_5) and (A_5').*

Question C.13. *Explain the following statement (both formally and intuitively): If f weakly dominates g, and moreover the belief μ is full-support (i.e., $\mu(\omega) > 0$ for all $\omega \in \Omega$), then $f \succ g$.*

Mixed act: take two acts $f, g \in \mathcal{F}$ such that $f(\omega) \in X$ and $g(\omega) \in X$, i.e., at each state a (sure) monetary outcome is paid. For some $\alpha \in (0, 1)$ we define the mixed act $(\alpha \times f, (1-\alpha) \times g)$ yields at each state $\omega \in \Omega$ the lottery $(\alpha \times f(\omega), (1-\alpha) \times g(\omega))$.

Question C.14 (Challenging)**.** *Consider the following three acts that pay different amounts depending on whether a Democrat or a Republican candidate wins the next american presidential elections.*

act	*Democrat*	*Republican*
f	*$10*	*$0*
g	*$0*	*$10*
h	*$5*	*$5*

Consider an individual with preferences over acts that have a SEU representation. Note that we do not know the individual's beliefs over the set of states. Prove formally and provide intuition for the following statements:

(a) If the individual is risk-seeking, then there exists some $\alpha \in (0, 1)$ such that the mixed act $(\alpha \times f, (1-\alpha) \times g)$ strictly dominates h.

(b) If the individual is risk-averse, then there exists a belief μ such that h is a rational choice in the decision problem $A = \{f, g, h\}$.

Problem C.2 (Very challenging)**.** *A forecaster is asked to report his subjective belief of the Democratic candidate winning the next american presidential elections, and reports a probability* $r \in [0, 1]$*, which of course we do not know if it is the same as his true belief. The forecaster is compensated depending on his reported belief and the realized state (i.e., based on what he said and who won). In particular, if the Democrat is elected then he wins $10,000 with probability* $1 - (1 - r)^2$*, whereas if the Republican is elected then wins $10,000 with probability* $1 - r^2$*.*

(a) Write each report $r \in [0, 1]$ *as an act.*

(b) If his preferences over acts have a SEU representation, prove that his unique optimal choice is to report his true belief. Which is the implication of this result?

Choice data: a dataset is a finite collection of pairs $(A_k, C_k)_{k=1}^K$ where $A_k \subseteq X$ is a decision problem and $C_k \subseteq A_k$ are the choices that are chosen by the individual when facing A_k.

Rationalizing: a preference relation $\succeq$ is said to rationalize the dataset $(A_k, C_k)_{k=1}^K$ whenever, $x \succeq y$ for all $x \in C_k$ and all $y \in A_k$. Importantly, rationalizing a dataset does not mean that we conclude that the individual is necessarily rational, or that we can conclusively find the individual's utility function. Instead it means that, *based on the data that we have available, we cannot reject the hypothesis that the individual behaves as if she chooses rationally given a preference relation with certain properties.*

Question C.15. (Allais Paradox). *Consider the following lotteries* $p_1 = (0.25 \times 3000, 0.75 \times 0)$*,* $p_2 = (0.2 \times 4000, 0.8 \times 0)$*,* $q_1 = (1 \times 3000)$ *and* $q_2 = (0.8 \times 4000, 0.2 \times 0)$*. An experimental subject chooses* p_2 *from* $\{p_1, p_2\}$ *and* q_1 *from* $\{q_1, q_2\}$*. Can this dataset be rationalized if the subject has vNM preferences? Explain.*

Question C.16. (Ellsberg Paradox). *Consider an urn containing three balls. An experimental subject is told that exactly one ball is red and the remaining two balls are either (i) both black, or (ii) one is black and the other one is yellow, or (iii) both yellow. A ball will be randomly drawn from the earn. Consider the following alternatives:*

f_1 : *if the ball is red you get 10 Euros*
f_2 : *if the ball is black you get 10 Euros*
g_1 : *if the ball is red or yellow you get 10 Euros*
g_2 : *if the ball is black or yellow you get 10 Euros*

First, define the states and write these alternatives as acts. Suppose that the subject chooses from f_1 *from* $\{f_1, f_2\}$ *and* g_2 *from* $\{g_1, g_2\}$*. Can this dataset be rationalized if the subject has AA preferences? Explain.*

D Strategic-form games

General aim: to introduce strategic uncertainty.

Question D.1. *What is strategic uncertainty? Is it objective or subjective?*

Strategic-form game: a tuple $(I, (A_i)_{i \in I}, (o_i)_{i \in I})$, where $I = \{\text{Ann } (a), \text{ Bob } (b)\}$ is the finite set of players, A_i is player i's finite set of actions, and $o_i : A_a \times A_b \to X$ induces an outcome for each player i and for each action profile (a_a, a_b). For simplicity, we will continue assuming that outcomes are monetary payoffs, i.e., $X \subseteq \mathbb{R}$. Hence, each action profile (a_a, a_b) yields a pair $\big(o_a(a_a, a_b), o_b(a_a, a_b)\big) \in \mathbb{R}^2$ of monetary payoffs.

Example D.1. Consider the following strategic form game, where $A_a = \{T, M, B\}$ and $A_b = \{L, R\}$.

	L	R
T	\$10 , \$5	\$0 , \$0
M	\$0 , \$5	\$10 , \$0
B	\$5 , \$0	\$5 , \$5

Each action profile yields a pair of monetary payoffs. The first number refers to Ann's payoff and the second one to Bob's. For instance, $\big(o_a(M, L), o_b(M, L)\big) = (\$0, \$10)$. ◁

Games as decision problems under subjective uncertainty: from player i's point of view, each of the opponent's actions $a_j \in A_j$ can be seen as a state, and each own action $a_i \in A_i$ can be seen as an act that yields a pair of payoffs at each state. In this sense, a game can be seen as a decision problem under subjective uncertainty.

Question D.2. *In the game from Example D.1, write each action of each player as an act.*

Preferences: Each player $i \in I$ has AA preferences $\succeq_i$ over the set of acts. Recall that a SEU representation of preferences over acts consists of a vNM utility function (over pairs of monetary payoffs) and a belief (over the opponent's actions).

Question D.3. *In the game of Example D.1, assume that both players are selfish. Then, provide three vNM utility representations, one where they are both risk-neutral, one where they are both risk-averse and one where they are both risk-seeking. Do the three utility functions represent the same preference ordering over action profiles?*

Question D.4. *Provide a utility function of Ann assuming that she is inequity-averse in the game of Example D.1?*

Subjective Expected Utility: it is defined analogously to our models of decision theory under subjective uncertainty. Namely, given a vNM utility function u_i and a belief μ_i, player i's SEU from an action a_i is equal to

$$\mathbb{E}_{\mu_i}(u_i(a_i)) := \sum_{a_j \in A_j} \mu_i(a_j) u_i(a_i, a_j).$$

Question D.5. *Explain the following statement: In the previous formula, we have simplified notation, by setting* $u_i(a_i, a_j) := u_i\big(o_i(a_i, a_j), o_j(a_i, a_j)\big)$.

RATIONALITY: given a belief μ_i, the action a_i of player i is rational if

$$\mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i'))$$

for all $a_i' \in A_i$.

Question D.6. *Using each of the three vNM utility functions from Question D.3 separately, find the actions of Ann that can be rationally be played for some belief in the game of Example D.1.*

METHODOLOGICAL APPROACH: it is not necessarily the case that others (i.e., the opponent or even the game-theorist who studies the game) can observe the vNM utility functions or the the beliefs of the players. So, we will distinguish two cases:

- ORDINAL GAMES: Only the ordinal preferences for allocations of monetary payoffs are commonly known.
- CARDINAL GAMES: The vNM preferences for lotteries over allocations of monetary payoffs are also commonly known. Of course, from the vNM preferences one can also deduce the ordinal preferences.

In both cases, we assume that it is also commonly known that the preferences satisfy (A_5') strict AA monotonicity. In both cases, we will assume that others do not know anything about the player's beliefs.

SOLUTION CONCEPT: it is a blackbox that takes as input the structure of the game (i.e., $(I, (A_i)_{i \in I}, (u_i)_{i \in I})$ and returns a collection of action profiles (i.e., a subset of $A_a \times A_b$) as prediction of what could be played. Importantly the prediction does not need to be unique. Solution concepts that we study in this course are classified as follows:

	ELIMINATION	EQUILIBRIUM
ORDINAL GAMES	Iterated Strict/Weak/Börgers Dominance	Nash Equilibrium
CARDINAL GAMES	Iterated Strict Dominance	Mixed Nash equilibrium

The row classification distinguishes games on the basis of the information that is available on the players' preferences, i.e., based on whether u_i is an ordinal or a vNM utility function. So, the difference here is simply on the input side of the solution concept. The column classification distinguishes solution concepts on the basis of the players' strategic reasoning. So, the difference here lies in what happens inside the blackbox. All solution concepts assume that players are rational, and differ in the additional assumptions that they make.

D.1 Ordinal games

ORDINAL UTILITY FUNCTION: the only information that we have is the players' preference over allocations of payoff pairs, or equivalently over $A_a \times A_b$ (see Question D.5). These preferences are represented by an ordinal utility function $u_i : A_a \times A_b \to \mathbb{R}$.

ELIMINATION SOLUTION CONCEPTS: If we only know the ordinal preferences of the players (but not their respective vNM preferences), we can often not say much about the rational actions (see Question D.6). So our approach will be to start eliminating actions that are definitely not rational.

There are different ways to do this, corresponding to different solution concepts. All of them utilize notions of dominance (see part on decision theory under uncertainty).

Strict Dominance: The action $a_i \in A_i$ strictly dominates the action $a_i' \in A_i$ in the game $A_i \times A_j$, whenever

$$u_i(a_i, a_j) > u_i(a_i', a_j)$$

for all $a_j \in A_j$. The idea a_i is strictly preferred to a_i' irrespective of what the opponent does. Note that dominance is a notion which is relative to the game $A_i \times A_j$. Whenever it is obvious which game we are referring to, we will omit explicit reference. Otherwise, we will have to explicitly mention it. If there is an action a_i that strictly dominates a_i', we say that a_i' is strictly dominated. If there is an action a_i that strictly dominates every other a_i', we say that a_i is strictly dominant.

Question D.7. (Public good game). *Ann and Bob are initially endowed with 10 Euros each. Each of them independently chooses an amount from the set* $\{0, 1, \ldots, 10\}$ *to put into a common account. The remaining of their endowment will go into a private account. The public account has an interest rate of 50%, i.e., the total amount that goes into the public account is multiplies by 1.5. On the other hand, the private accounts do not have any interest. Each individual's total payoff will be equal to the amount in their private account plus half the amount in the public account. Assuming that both are selfish, is there a strictly dominant action for each of them? Explain formally and intuitively.*

Question D.8. *Assume that a player's preferences satisfy* (A_5') *strict AA monotonicity. Then, explain the following statements: (a) a strictly dominated action is definitely not rational, (b) a strictly dominant action is the only rational action.*

Iterated Strict Dominance (ISD): For each step $k \geq 0$ and each player $i \in I$:

$$\begin{aligned}
S_i^0 &:= A_i \\
S_i^1 &:= \{a_i \in S_i^0 : a_i \text{ is not strictly dominated in } S_i^0 \times S_j^0\} \\
S_i^2 &:= \{a_i \in S_i^1 : a_i \text{ is not strictly dominated in } S_i^1 \times S_j^1\} \\
&\vdots \\
S_i^k &:= \{a_i \in S_i^{k-1} : a_i \text{ is not strictly dominated in } S_i^{k-1} \times S_j^{k-1}\} \\
&\vdots
\end{aligned}$$

Then, the actions of player i that survive ISD are those in $S_i := \bigcap_{k=0}^{\infty} S_i^k$. The predictions of ISD are the action profiles in $ISD := S_a \times S_b$.

Weak Dominance: The action $a_i \in A_i$ weakly dominates the action $a_i' \in A_i$ in the game $A_i \times A_j$, whenever

$$u_i(a_i, a_j) \geq u_i(a_i', a_j)$$

for all $a_j \in A_j$, with at least one of the inequalities being strict. If there is an action a_i that weakly dominates a_i', we say that a_i' is weakly dominated. If there is an action a_i that weakly dominates every other a_i', we say that a_i is weakly dominant.

Question D.9. (Second-price auction). *Ann and Bob participate in an auction for painting. Each of them has a budget of \$100k. Ann's private value for the object is \$40k and Bob's private value is \$30k. The rules of the auction are as follows: (a) each participant can bid multiples of \$1k,* *(b) the winner is the participant with the highest bid, and in case of a tie Ann is the winner, (c) the winner pays a price equal to the bid of the other participant (i.e., the second price). Explain the following statement: it is a weakly dominant action for Ann to bid \$40k.*

Question D.10. *Assume that a player's preferences satisfy (A_5) AA monotonicity and (A_5') strict AA monotonicity. Then, explain the following statements: (a) a strictly dominated action is definitely not rational if the player has full-support beliefs, (b) a weakly dominant action is rational.*

Iterated Weak Dominance (IWD): For each step $k \geq 0$ and each player $i \in I$:

$$
\begin{aligned}
W_i^0 &:= A_i \\
W_i^1 &:= \{a_i \in W_i^0 : a_i \text{ is not weakly dominated in } W_i^0 \times W_j^0\} \\
W_i^2 &:= \{a_i \in W_i^1 : a_i \text{ is not weakly dominated in } W_i^1 \times W_j^1\} \\
&\vdots \\
W_i^k &:= \{a_i \in W_i^{k-1} : a_i \text{ is not weakly dominated in } W_i^{k-1} \times W_j^{k-1}\} \\
&\vdots
\end{aligned}
$$

Then, the actions of player i that survive IWD are those in $W_i := \bigcap_{k=0}^{\infty} W_i^k$. The predictions of IWD are the action profiles in $IWD := W_a \times W_b$.

Börgers Dominance: The action $a_i \in A_i$ is Börgers dominated in the game $A_i \times A_j$, whenever for every nonempty $A_j' \subseteq A_j$ there exists some $a_i' \in A_i$ (which may depend on A_j') such that a_i' weakly dominates a_i in $A_i \times A_j'$.

Theorem D.1. *For an action $a_i \in A_i$ the following statements are equivalent:*

(a) a_i is Börgers dominated in $A_i \times A_j$.

(b) There is no vNM utility function (with the same preference ordering over $A_i \times A_j$) and belief μ_i such that a_i is rational given μ_i.

Problem D.1. *In the following game, assume that both players are selfish.*

	L	R
T	\$10 , \$5	\$0 , \$0
M	\$0 , \$5	\$1 , \$0
B	\$9 , \$0	\$0 , \$5

Show that action B is Börgers dominated, while M is not Börgers dominated. Then, prove that there is no vNM utility function u_a and no belief μ_a such that B is rational in $\{T, M, B\} \times \{L, R\}$, but there exists a vNM utility function u_a and a belief μ_a such that M is rational in $\{T, M, B\} \times \{L, R\}$.

Iterated Börgers Dominance (IBD): For each step $k \geq 0$ and each player $i \in I$:

$$\begin{aligned} B_i^0 &:= A_i \\ B_i^1 &:= \{a_i \in B_i^0 : a_i \text{ is not Börgers dominated in } B_i^0 \times B_j^0\} \\ B_i^2 &:= \{a_i \in B_i^1 : a_i \text{ is not Börgers dominated in } B_i^1 \times B_j^1\} \\ &\vdots \\ B_i^k &:= \{a_i \in B_i^{k-1} : a_i \text{ is not Börgers dominated in } B_i^{k-1} \times B_j^{k-1}\} \\ &\vdots \end{aligned}$$

Then, the actions of player i that survive IBD are those in $B_i := \bigcap_{k=0}^{\infty} B_i^k$. The predictions of IBD are the action profiles in $IBD := B_a \times B_b$.

Question D.11. *Explain the following statement: If a player has a dominant strategy, we do not impose any assumption about her strategic reasoning. On the other hand, solution concepts that are based on iteratively eliminating dominated strategies, make implicit assumptions about the players' strategic reasoning.*

Question D.12. *Prove formally that $IWD \subseteq IBD \subseteq ISD$. Explain what it says.*

Question D.13. (Guessing game). *This is one of the very few instances where we will have games with more than two players. There are n experimental subjects, and each of them is asked to chooses an number from the set $\{1, 2, \ldots, 100\}$. We then compute the average, and whoever is closer to 2/3 of the average wins a prize of \$100. If there are several winners, they split the prize. If everybody is selfish, IWD predicts that everybody will choose 1. Explain why this is the case.*

Question D.14. *Experimental findings consistently show that actual subjects in the guessing game choose mostly numbers above 50, and often even above 90. Where do you think we can attribute this behavior?*

Equilibrium solution concepts: the nice thing with elimination solution concepts is that each player can run the algorithm in their head and come up with the solution without needing any prior information about the opponent's play. This makes them very reasonable but quite conservative. On the other hand equilibrium concepts, implicitly assume that each player can correctly predict the opponent's action, and respond rationally to it. This can be justified as the steady state of some learning process.

Best response: the action $a_i \in A_i$ is a best response to $a_j \in A_j$ if

$$u_i(a_i, a_j) \geq u_i(a_i', a_j)$$

for all $a_i' \in A_i$. Importantly, we do not compare a_i with a_i' for every action of the opponent, but only for a_j. There may exist multiple best responses to a_j. The set of all best responses to a_j will be denoted by $BR_i(a_j)$. So, if a_i is a best response to a_j we write $a_i \in BR_i(a_j)$.

Question D.15. *Explain the following statements: (a) if a_i is a best response to some a_j then it is not strictly dominated, but it may still be weakly dominated, (b) if a_i is strictly dominant then it is the only best response to each $a_j \in A_j$, (c) if a_i is weakly dominant then it is the one of the (perhaps multiple) best responses to each $a_j \in A_j$.*

Nash Equilibrium: the action profile (a_a, a_b) is a Nash equilibrium if $a_i \in BR_i(a_j)$ for both $i \in I$. That is, none of the players has an incentive to unilaterally deviate from this action profile. The set of all Nash Equilibria is denoted by $NE \subseteq A_a \times A_b$.

Question D.16. *There are two firms that serve a market, one owned by Ann and one by Bob. The two firms simultaneously choose a price each (p_a and p_b respectively) from the set $\{1, 2, 3, 4\}$. If one of the two firms charges a lower price than the opponent, this firm will serve the entire market, i.e., the market price will be $p = \min\{p_a, p_b\}$ and the firm with the lowest price will sell $Q(p) := 16 - 4p$ units of the product whereas the other firm will sell 0 units. If the two firms charge the same price, the market price is $p = p_a = p_b$ and they share the market, i.e., each of the two firms sells $Q(p)/2$ units. Suppose that the cost per unit for Ann is equal to $c_a = 1$ and for Bob is equal to $c_b = 2$. Both firms are selfish in the sense that they want to maximize their own profits. Find the Nash Equilibria of this game.*

Question D.17. *Take the following three very well-known games, assuming that the utilities are vNM. Nevertheless, only ordinal the preferences over action profiles are commonly known.*

	D	C
D	2 , 2	7 , 0
C	0 , 7	5 , 5

Prisoner's Dilemma

	O	B
O	3 , 1	0 , 0
B	0 , 0	1 , 3

Battle of the Sexes

	H	T
H	1 , -1	-1 , 1
T	-1 , 1	1 , -1

Matching Pennies

Find the Nash Equilibria in each of the corresponding ordinal games.

Problem D.2. (First-price auction). *Ann and Bob participate in an auction for a painting. Each of them has a budget of \$100k. Ann's private value for the object is \$40k and Bob's private value is \$30k. The rules of the auction are as follows: (a) each participant can bid multiples of \$1k, (b) the winner is the participant with the highest bid, and in case of a tie Ann is the winner, (c) the winner pays a price equal to his/her own bid (i.e., the first price). Is it a Nash Equilibrium if both players bid their own private value? Find a Nash Equilibrium where one of the players bids above his/her private value.*

Question D.18. *Explain the following statement: Every Nash Equilibrium survives ISD but not necessarily the other way around.*

Question D.19. *The previous statement does not extend to IWD, i.e., a Nash Equilibrium may not survive IWD. Provide an example where a Nash Equilibrium is eliminated by IWD.*

D.2 Cardinal games

vNM utility function: we have information about the players' preferences for lotteries over allocations of payoffs. These preferences are represented by a vNM utility function $u_i : A_a \times A_b \to \mathbb{R}$.

Mixed action: The fact that we can now attach a vNM utility to each payoff allocation means that we can introduce probabilities and take expectations. A mixed action σ_i is a probability distribution over the player's actions, which attaches probability $\sigma_i(a_i)$ to each $a_i \in A_i$. This probability distribution is chosen by player i herself. The interpretation of a mixed action is that the player delegates her action to a randomizing device, whose probabilities are objectively known. Obviously, an action $a_i \in A_i$ can be seen as a (degenerate) mixed strategy σ_i such that $\sigma_i(a_i) = 1$. The set of all mixed actions of player i is denoted by $\mathcal{L}(A_i)$.

Expected utility from a mixed action: Following the interpretation that from i's point of view the game can be seen as a decision problem under subjective uncertainty, a mixed action is then seen as a mixed act. This means that a mixed action induces an (objective) expected utility for each action of the opponent. Formally, for a mixed action $\sigma_i \in \mathcal{L}(A_i)$ and each action $a_j \in A_j$, player i's expected utility is given by

$$u_i(\sigma_i, a_j) := \sum_{a_i \in A_i} \sigma_i(a_i) u_i(a_i, a_j).$$

Question D.20. *Using each of the three vNM utility functions from Question D.3 separately, find the Ann's expected utilities from a mixed action σ_a such that puts probability 1/2 to T and probability 1/4 to M and 1/4 to B.*

Elimination solution concepts: even though we know the vNM preferences of the players, we still do not know their beliefs about the opponent's actions (i.e., we still do not know the full AA preferences). Hence, we cannot deduce their actual rational actions. So our approach will be to start eliminating actions that are definitely not rational, similarly to the ordinal case. The only difference is that we now have additional information (i.e., the vNM preferences, rather than just the ordinal preferences). Hence, we can implement similar elimination procedures as in the ordinal case, taking into account this additional information that we now have.

Strict Dominance: A mixed action $\sigma_i \in \mathcal{L}(A_i)$ strictly dominates the action $a_i \in A_i$ in the game $A_i \times A_j$ whenever

$$u_i(\sigma_i, a_j) > u_i(a_i, a_j)$$

for all $a_j \in A_j$. If there exists a mixed action σ_i that strictly dominates a_i then we say that a_i is strictly dominated.

Question D.21 (Challenging)**.** *Using each of the three vNM utility functions from Question D.3 separately, find the actions of Ann that are strictly dominated in the game of Example D.1.*

Theorem D.2. *For an action $a_i \in A_i$, the following two statements are equivalent:*

(a) There exists some mixed action σ_i that strictly dominates a_i.

(b) There is no belief μ_i such that a_i is rational given μ_i.

Question D.22. *Rewrite the previous theorem in the following form: the negation of (a) is equivalent to the negation of (b).*

Question D.23. *Using Theorem D.2, explain that Question D.6 is equivalent to Question D.21.*

Question D.24. *Explain the following statement: If an action a_i is strictly dominated in an ordinal game, then it is also strictly dominated in every cardinal game that agrees with the preference ordering over $A_i \times A_j$. The converse is not necessarily true. This means that when we eliminate a strictly dominated action in the ordinal game, we are sure that this action is not rational in the cardinal game, but at the same time we may keep too many actions (i.e., we may keep actions that are not rational for any belief in the cardinal game).*

Question D.25. *Explain the following statement: The same holds true for Börgers dominance, i.e., when we eliminate a Börgers dominated action in the ordinal game, we are sure that this action is not rational in the cardinal game, but at the same time we may keep too many actions (i.e., we may keep actions that are not rational for any belief in the cardinal game).*

ITERATED STRICT DOMINANCE (ISD): The definition is identical to the one for ordinal games, except for the fact that the notion of strict dominance is now adjusted to allow for mixed strategies. For each step $k \geq 0$ and each player $i \in I$:

$$
\begin{aligned}
S_i^0 &:= A_i \\
S_i^1 &:= \{a_i \in S_i^0 : a_i \text{ is not strictly dominated in } S_i^0 \times S_j^0\} \\
S_i^2 &:= \{a_i \in S_i^1 : a_i \text{ is not strictly dominated in } S_i^1 \times S_j^1\} \\
&\vdots \\
S_i^k &:= \{a_i \in S_i^{k-1} : a_i \text{ is not strictly dominated in } S_i^{k-1} \times S_j^{k-1}\} \\
&\vdots
\end{aligned}
$$

Then, the actions of player i that survive ISD are those in $S_i := \bigcap_{k=0}^{\infty} S_i^k$. The predictions of ISD are the action profiles in $ISD := S_a \times S_b$.

Problem D.3. (TRAVELLER'S DILEMMA). *An airline loses two identical suitcases belonging to two different risk-neutral travellers. The airline manager explains that the airline is liable for a maximum of \$6 per suitcase, and in order to determine the actual value of the suitcase he asks them to write down the amount of their value at no less than \$2 and no larger than \$6 (only whole amounts are allowed). If both write down the same number, he will treat that number as the true value of both suitcases and reimburse both travellers that amount. If one writes down a smaller number than the other, this smaller number will be taken as the true value, and both travellers will receive that amount along with a bonus/malus: \$2 extra will be paid to the traveler who wrote down the lower value and a \$2 deduction will be taken from the person who wrote down the higher amount. Find the action profiles that survive ISD in this game.*

EQUILIBRIUM SOLUTION CONCEPTS: the notion of equilibrium is entirely analogous to the one in ordinal games. The only difference is that now we allow for mixed actions.

BEST RESPONSE: the action $a_i \in A_i$ is a best response to $\sigma_j \in \mathcal{L}(A_j)$ if

$$\mathbb{E}_{\sigma_j}(u_i(a_i)) \geq \mathbb{E}_{\sigma_j}(u_i(a_i'))$$

for all $a_i' \in A_i$. A mixed action σ_i is a best response to σ_j if and only if every action in the support of σ_i is also a best response to σ_j (see Question D.26 below). The set of all best responses to σ_j will be denoted by $BR_i(\sigma_j)$.

Question D.26. *Explain why the following statement holds: $\sigma_i \in BR_i(\sigma_j)$, if and only if, $a_j \in BR_i(\sigma_j)$ for every $a_i \in A_i$ such that $\sigma_i(a_i) > 0$.*

MIXED NASH EQUILIBRIUM: the action profile (σ_a, σ_b) is a Nash equilibrium if $\sigma_i \in BR_i(\sigma_j)$ for both $i \in I$. The set of all Nash Equilibria is denoted by $NE \subseteq \mathcal{L}(A_a) \times \mathcal{L}(A_b)$.

Theorem D.3. *At least one mixed Nash Equilibrium always exists.*

Question D.27. *Explain the following statement: if (σ_i, σ_j) is a Mixed Nash Equilibrium, then every $a_i \in A_i$ such that $\sigma_i(a_i) > 0$ survives ISD.*

Question D.28. *Find the Mixed Nash Equilibria of "Battle of the Sexes" and "Matching Pennies" from Question D.17.*

Problem D.4. *Find the Mixed Nash Equilibria in the following game:*

	L	*M*	*R*
T	*7 , 1*	*0 , 1*	*6 , 0*
M	*0 , 1*	*10 , 1*	*4 , 0*
B	*4 , 0*	*3 , 0*	*5 , 5*

Question D.29. *Mixed Nash Equilibrium is often tested using aggregate data. This means that we observe n independent plays of the same game, and we check if the empirical frequency of play for the role of player a versus the empirical frequency of play for the role of player b form a Mixed Nash Equilibrium. What is the implicit underlying assumption that we make here if all observations come from the same pair of subjects playing against each other? Which is the additional implicit assumption that we make when the different observations come from different pairs of subjects?*

Question D.30. *Explain the following statements intuitively: in a cardinal game, if we take a strictly increasing linear transformation of the vNM utilities, both ISD and NE remain unchanged. On the other hand in an ordinal game, if we take any strictly increasing transformation (not necessarily linear) of the ordinal utility functions, ISD, IWD, IBD and NE remain all unchanged. Relate your answers to the ones of Questions B.8 and C.5.*

E Extensive-form games

General aim: to introduce time into our model.

Game tree: a finite (directed) graph $(\bar{H}, \mathcal{E})$, where $\bar{H}$ is the set of all nodes and $\mathcal{E} := \{(h, h') \in \bar{H} \times \bar{H} : h' \text{ is a direct successor of } h\}$. We say that h'' is a successor of h if there is a directed path from h to h'', i.e., if there exists a sequence of nodes $(h_1, \ldots, h_N)$ such that (a) $h_1 = h$, (b) $h_N = h''$, and (c) $(h_n, h_{n+1}) \in \mathcal{E}$ for all $n = 1, \ldots, N-1$. The set of weak successors of h contains all its successors plus h itself. Whenever h' is a direct successor of h, we say that h is a direct predecessor of h'. The root $h_0 \in \bar{H}$ is the only node that does not have any direct predecessor. Every other node has exactly one direct predecessor. Those nodes that do not have any direct successor are called terminal nodes and are denoted by $Z \subseteq \bar{H}$. All non-terminal nodes are denoted by $H := \bar{H} \setminus Z$.

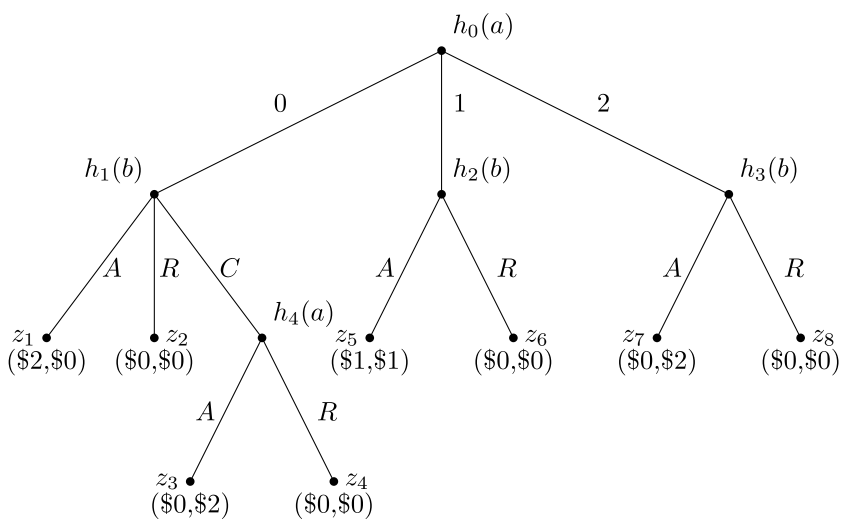
Active players: The set of players is still assumed to be $I = \{\text{Ann } (a), \text{ Bob } (b)\}$. There is a function $p : H \to I$, assigning a player $p(h)$ to every non-terminal node. This is the player who decides to which direct successor of h we will move if h is reached. The set of nodes where player i is active is denoted by H_i.

Actions: At a non-terminal node $h \in H_i$, the active player's set of actions are identified by the direct successors, and are denoted by $A_i(h)$. That is, each $a_i \in A_i(h)$ leads to some $h' \in \bar{H}$ such that $(h, h') \in \mathcal{E}$. For simplicity, we assume that $A_i(h)$ has at least two actions.

Outcomes: There is an outcome function $o_i : Z \to X$ for each player $i \in I$. For simplicity, we keep assuming that $X \subseteq \mathbb{R}$.

Preferences and Utilities: Each player $i \in I$ is assumed to have preferences $\succeq_i$ over allocations of playoffs $(o_a(z), o_b(z)) \in \mathbb{R}^2$. Similarly to strategic-form games. For the most part, we will consider ordinal preferences over allocations of payoffs. Hence, for each player $i \in I$ there is an ordinal utility function $u_i : Z \to \mathbb{R}$ representing $\succeq_i$. We will say that a game is without ties if, for every player $i \in I$ and every two terminal nodes $z, z' \in Z$ it is the case that $u_i(z) \neq u_i(z')$.

Example E.1. Ann starts by proposing a way to split of \$2. She can offer \$0 or \$1 or \$2 to Bob. If she offers \$1 or \$2, Bob can either accept or reject Ann's proposal. If he rejects they both receive \$0. If on the other hand she offers \$0 to Bob, then he can accept or reject or counteroffer. In the first two cases, the game proceeds identically as with



other two offers, whereas in case of a counteroffer, the choice goes back to Ann who either accepts to give all the money to Bob or rejects and they both get nothing. Both players are selfish. ◁

Question E.1. *Identify the different elements of an extensive-form game (i.e., nodes, active players, actions, outcomes, preferences) in the game of Example E.1.*

STRATEGIES: a strategy is a complete plan of action that specifies what the player does at every node that she is active. Formally, a strategy s_i assigns an action $s_i(h) \in A_i(h)$ to each $h \in H_i$. The set of all strategies is denoted by $S_i := \times_{h \in H_i} A_i(h)$. A strategy is typically assumed to be devised by the player at the beginning of the game, i.e., before the first move at h_0 takes place.

Question E.2. *Which are the strategies of each of the two players in the game of Example E.1?*

Question E.3 (Challenging, but important)**.** *Sometimes a strategy s_i prescribes an action at some node $h \in H_i$ that rules out a node $h' \in H_i$. Nevertheless, s_i still prescribes an action to h'. This is done so that player i is prepared to act following possible mistakes that she may make herself. The importance of being prepared for some mistakes will become clear later on. Explain the previous statement. Identify such a strategy in the game of Example E.1.*

NODES CONSISTENT WITH A STRATEGY: For a strategy $s_i \in S_i$, the non-terminal nodes that can be reached are $H(s_i)$. By $H_i(s_i) := H(s_i) \cap H_i$ we denote the ones where i is active and can be reached by s_i.

Question E.4. *In the game of Example E.1, for each strategy $s_i \in S_i$ of each player $i \in I$ find the sets $H(s_i)$ and $H_i(s_i)$.*

Question E.5. *Explain the following statement: Each strategy profile uniquely identifies a terminal node. Hence, from the preferences over the set of terminal nodes, we obtain preferences over the set of strategy profiles, $S_a \times S_b$. However, for each terminal node we cannot necessarily identify which strategy profile has been played. Illustrate your answer in the game of Example E.1.*

INFORMATION: we distinguish extensive-form games on the basis of what players observe about past actions that have already taken place. There are two classes of games we consider:

- PERFECT INFORMATION GAMES: The active player at each node knows which are the nodes that have been previously visited, and can therefore deduce what exactly has been previously played.
- IMPERFECT INFORMATION GAMES: The active player at some node does not know which are the nodes that have been previously visited. The interpretation is that some of the past actions have not been observed by the player.

For imperfect information games, we will later on slightly refine the notion of a strategy.

SOLUTION CONCEPT: similarly to strategic-form games, it is a blackbox that takes as input the structure of the game (i.e., $(I, H, Z, p, (\succeq_i)_{i \in I})$) and returns as output either a collection of strategy profiles (i.e., a subset of $S_a \times S_b$). The solution concepts that we will study are classified as follows (both for perfect information games):

	ELIMINATION	EQUILIBRIUM
BEGINNING OF THE GAME	–	Nash Equilibrium
THROUGHOUT THE WHOLE GAME	Backward Induction	Subgame Perfect Equilibrium

The row classification distinguishes solution concepts on the basis of whether each player revisits the rationality of her strategies at every node where she is active. The column classification is the same as in the case of strategic-form games. For imperfect information games, we will focus only on Nash Equilibrium and Subgame Perfect Equilibrium.

Question E.6. *Sometimes we only consider the outcomes (i.e., the terminal nodes) that the solution concept predicts rather than the strategy profiles. This is because, it is difficult to obtain data on the strategies that players choose. What does this mean? What is the strategy method in a lab experiment?*

E.1 Perfect information games

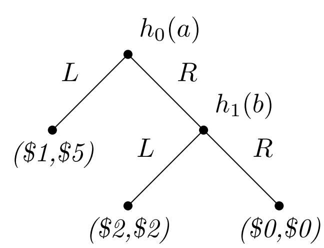
CORRESPONDING STRATEGIC FORM: it is the (ordinal) game $(I, (S_i)_{i\in I}, (\succeq_i)_{i\in I})$ that is obtained by assuming that each of the two players chooses a strategy at the beginning of the game, and then they simply follow these strategies (without ever reconsidering them throughout the game) until a terminal node is reached.

Question E.7. *Write the corresponding strategic form of the game in Example E.1.*

NASH EQUILIBRIUM: it is the Nash Equilibrium of the corresponding strategic form of the game.

Question E.8. *Write the corresponding strategic form and find the Nash Equilibria of the game in Example E.1. Are there Nash Equilibria that do not make intuitive sense? Explain.*

Question E.9. *Consider the following very well-known game:*



Chain-store paradox

Write the corresponding strategic form and find the Nash Equilibria. Are there Nash Equilibria that do not make intuitive sense?

CORRESPONDING STRATEGIC FORM AT ARBITRARY NODE: once a node $h \in H$ has been reached, it becomes common knowledge that certain strategies have not been chosen. The strategies of player i that are consistent with h being reached are denoted by $S_i(h)$. Then, the corresponding strategic form of the game at some $h \in H$ is the reduced game $(I, (S_i(h))_{i\in I}, (\succeq_i)_{i\in I})$.

Problem E.1. *Explain the following statements: If h is a direct predecessor of h' then $S_i(h') \subseteq S_i(h)$ for all $i \in I$, with equality holding if $p(h) \neq i$.*

Backward Induction (BI): at each step k, it eliminates strategies at each node $h \in H_i$ and for every $i \in I$:

$$\begin{aligned}
S_i^0(h) &:= S_i(h) \\
S_i^1(h) &:= \{s_i \in S_i^0(h) : s_i \text{ is not strictly dominated in } S_i^0(h') \times S_j^0(h') \\
&\qquad \text{for any } h' \in H_i(s_i) \text{ that weakly succeeds } h\} \\
S_i^2(h) &:= \{s_i \in S_i^1(h) : s_i \text{ is not strictly dominated in } S_i^1(h') \times S_j^1(h') \\
&\qquad \text{for any } h' \in H_i(s_i) \text{ that weakly succeeds } h\} \\
&\vdots \\
S_i^k(h) &:= \{s_i \in S_i^{k-1}(h) : s_i \text{ is not strictly dominated in } S_i^{k-1}(h') \times S_j^{k-1}(h') \\
&\qquad \text{for any } h' \in H_i(s_i) \text{ that weakly succeeds } h\} \\
&\vdots
\end{aligned}$$

Then, the strategies of player i that survive BI are those in $S_i^\infty := \bigcap_{k=0}^\infty S_i^k(h_0)$. The predictions of BI are the strategy profiles in $BI := S_a^\infty \times S_b^\infty$.

Theorem E.1. *If the game has no ties, there is a unique strategy profile surviving BI.*

Question E.10. *Apply Backward Induction in the game of Example E.1.*

Question E.11. *The Backward Induction algorithm described above is equivalent to the following procedure, which can be directly applied on the extensive form of the game. The nodes that are followed only by terminal nodes are called Nodes of Group 1. The nodes that are direct predecessors to Nodes of Group 1 are called Nodes of Group 2, and inductively the nodes that are direct predecessors to Nodes of Group k are called Nodes of Group $k+1$. Then, begin by eliminating the actions that are not optimal at the Nodes of Group 1. Then, eliminate the actions that are never optimal at the Nodes of Group 2, given that the Nodes of Group 1 have been deleted. Continue inductively. In the end take the strategy profiles that use only the remaining actions. Explain why this procedure is equivalent to the one defined above. Illustrate your answer in the context of Example E.1.*

Question E.12. *In the context of Example E.1 illustrate that $BI \nsubseteq NE$ and $NE \nsubseteq BI$. Provide intuition. Nevertheless, if the game does not have ties, it will be the case that $BI \subseteq NE$. Illustrate your answer in the context of Question E.9.*

Continuation strategies: Here it will become clear why strategies prescribe actions also at nodes that are precluded by previous own actions (see Question E.3). For a strategy $s_i \in S_i$, define the strategy $s_i^h \in S_i(h)$ as follows: at all nodes h' that precede h take $s_i^h(h')$ to be the action that leads to h, while at all other nodes h'' take $s_i^h(h'') = s_i(h'')$. Of course, if $h \in H(s_i)$ then $s_i^h = s_i$.

Question E.13. *In the game of Example E.1 take $s_a := 1R$. Then, which is $s_a^{h_4}$?*

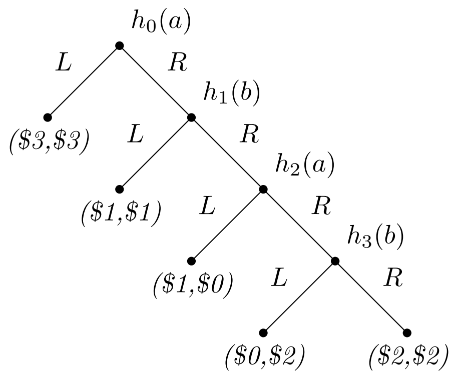
Subgame Perfect Equilibrium (SPE): it is a strategy profile which is a Nash Equilibrium at every $h \in H$. Formally, (s_a, s_b) is a Subgame Perfect Equilibrium, if (s_a^h, s_b^h) is a Nash Equilibrium in the game $S_a(h) \times S_b(h)$ for every $h \in H$. The set of Subgame Perfect Equilibria is denoted by $SPE \subseteq S_a \times S_b$. In perfect information games, at least one SPE always exists.

Question E.14. *Explain the following statement: a Subgame Perfect Equilibrium is a Nash Equilibrium which is robust to mistakes that players may make in the implementation of their strategies.*

Question E.15. *Compute the Subgame Perfect Equilibria in the games of Example E.1 and Question E.9.*

Theorem E.2. *A Subgame Perfect Equilibrium survives Backward Induction.*

Problem E.2. *Not all Nash Equilibria that survive Backward Induction are necessarily Subgame Perfect Equilibria. Illustrate the previous statement in the following example:*



In particular, find a Nash Equilibrium which survives Backward Induction, but is not a Subgame Perfect Equilibrium.

Question E.16. *Prove the following statement: In perfect information games without ties, the Subgame Perfect Equilibrium gives a unique prediction.*

Question E.17. *Consider the following well-known games:*

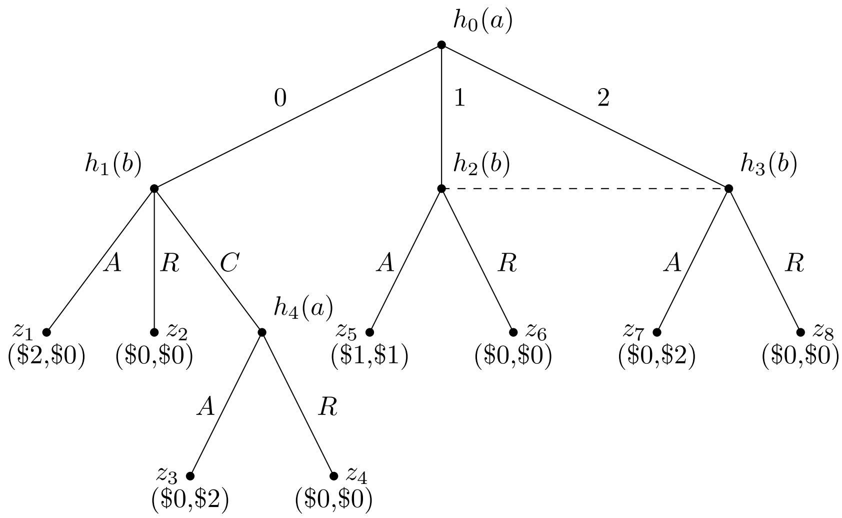
- Ultimatum game*: Ann proposes a split of \$10 into integer amounts between herself and Bob. If Bob accepts, the proposed allocation is implemented. If he rejects, they both receive \$0.*
- Trust game: *Ann chooses a split of \$10 into integer amounts between herself and Bob. The amount that Bob receives is doubled. Then, Bob decided how to split his (doubled) amount into integer amounts between himself and Ann.*

In the previous two games, find the Nash Equilibria and the Subgame Perfect Equilibria.

E.2 Imperfect information games

Information sets: we take non-terminal nodes that are controlled by i and we bundle them together. Intuitively, $I_i \subseteq H_i$ is an information set whenever the following holds: every time time some $h \in I_i$ is reached, player i does not know that h has been indeed reached, and instead considers possible all the nodes in I_i. As a consequence, if $h, h' \in I_i$ then $A_i(h) = A_i(h')$. Formally, $\mathcal{I}_i$ is the collection of all information sets of player i.

Example E.2. Consider the following variant of the game from Example E.1: everything remains the same as before, except for the fact that Bob does not observe



whether Ann has chosen 1 or 2 at h_0. Bob's failure to observe Ann's past action in these cases is modelled by bundling together h_2 and h_3 into one information set. This is graphically illustrated by connecting these two nodes with a dashed line. ◁

Question E.18. *Which are the information sets in the perfect information game of Example E.1?*

Question E.19. *Explain the following statement: whenever $h, h' \in I_i$ it must necessarily be the case that i chooses the same action at h and at h'.*

STRATEGIES: a strategy is still a complete plan of action that specifies what the player does at every node that she is active, subject to the restriction discussed in Question E.19. Formally, a strategy s_i assigns an action $s_i(h) \in A_i(h)$ to each $h \in H_i$, such that $s_i(h) = s_i(h')$ for all $h, h' \in I_i$ and all $I_i \in \mathcal{I}_i$. In this sense, s_i can be rewritten as a function that assigns an action $s_i(I_i)$ to each information set $I_i \in \mathcal{I}_i$. The set of all strategies is thus denoted by $S_i := \times_{I_i \in \mathcal{I}_i} A_i(I_i)$.

Question E.20. *Which are the strategies of each of the two players in the game of Example E.2?*

KNOWLEDGE: each information set I_i is identified by a subset of $S_a \times S_b$, i.e., by the strategy profiles that are consistent with reaching this information set. The same is true for every other relevant event in the game, e.g., player i choosing action $a_i \in A_i(h)$ is identified by the collection of all strategy profiles that prescribe a_i to h. Then, we say that an event E is known at the information set I_i, if the collection of profiles that identifies I_i is a subset of the collection of profiles that identifies the event E.

Problem E.3. *In the game of Example E.2, identify by means of a subset of $S_a \times S_b$ Bob's information set $\{h_2, h_3\}$. Similarly, identify the event that Ann has picked the action $R \in A_a(h_4)$.*

(a) Does Bob know at $\{h_2, h_3\}$ that Ann has picked R at h_4?

(b) Does Bob know at $\{h_2, h_3\}$ what Ann has played in the past? Explain using the formal way of representing knowledge.

PERFECT RECALL: each player remembers what she did in the past and what she knew in the past. Henceforth, we assume that the information sets satisfy the following conditions:

- Recall of past own actions: if h precedes h', and the corresponding information sets are $h \in I_i$ and $h' \in I_i$, then player i knows at h' which action she took herself at h.
- Recall of past own knowledge: if h precedes h', and the corresponding information sets are $h \in I_i$ and $h' \in I_i$, then player i knows at h' every event she knew at h'.

Question E.21. *Suppose that player i is absentminded, i.e., there are two succeeding nodes $h, h' \in I_i$. Then, show that recall of past actions is violated.*

Question E.22. *Provide an example where recall of past knowledge is violated.*

Subgame: it a reduced extensive-form game (obtained by deleting parts of the tree) which satisfies the following conditions: (a) it has a unique root $h \in H$, (b) if $h \in I_i$ belongs to the subgame, so does every other $h' \in I_i$, (c) if h belongs to the subgame, so does every succeeding (non-terminal or terminal) node. Every subgame is identified by its root.

Question E.23. *Which are the subgames in Example E.2?*

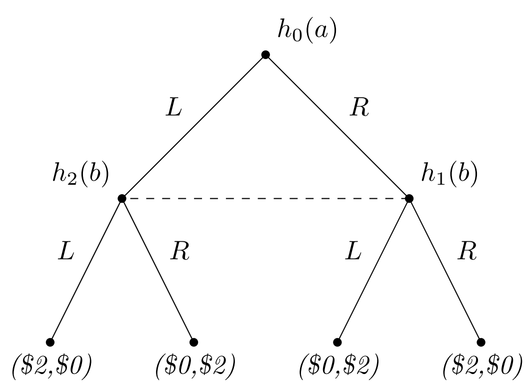
Corresponding strategic form at a subgame: the root h of a subgame has been reached, take the strategies of each player i that are consistent with h and define them by $S_i(h)$. Then, similarly to the perfect information case, the corresponding strategic form of the game at this subgame becomes $S_i(h) \times S_j(h)$.

Nash Equilibrium: identically to perfect information games, it is the Nash Equilibrium of the corresponding normal form at h_0.

Subgame Perfect Equilibrium: it is a strategy profile which is a Nash Equilibrium at every subgame. Formally, (s_a, s_b) is a Subgame Perfect Equilibrium, if (s_a^h, s_b^h) is a Nash Equilibrium in the game $S_a(h) \times S_b(h)$ for every node h that identifies some subgame, where the continuation strategy $s_i^h$is defined identically to the perfect information case. The set of Subgame Perfect Equilibria is again denoted by $SPE \subseteq S_a \times S_b$.

Question E.24. *Find the Nash Equilibria and the Subgame Perfect Equilibria in Example E.2.*

Problem E.4. *In imperfect information games, if we only know the ordinal utilities, we cannot always find a Subgame Perfect Equilibrium. Illustrate the previous statement in teh following example:*



In particular, show that this game does not have any Subgame Perfect Equilibrium.