Individual and Strategic Decisions (HDS)

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DECISION THEORY : Mathematical models

- One decision maker (DM)
- Certainty
- Reasoning capacity unlimited

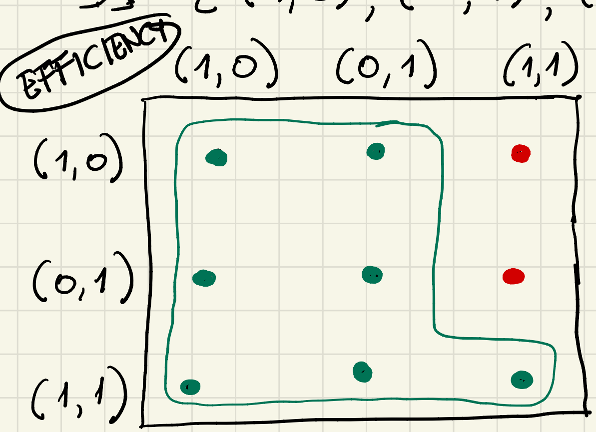
OUTCOMES $\underline{X}$ with elements x, y, z.

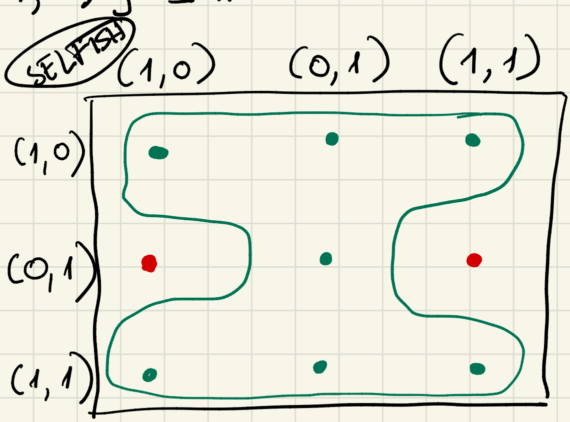
- Monetary payoffs $\underline{X} \subseteq \mathbb{R}$
- Bundles of goods $\underline{X} \subseteq \mathbb{R}^2$
  $x = (x_1, x_2)$
- Allocation of payoffs $\underline{X} \subseteq \mathbb{R}^4$
  $x = (x_1, x_2, x_3, x_4)$

PREFERENCES attitudes of the DM towards elements of $\underline{X}$.

weak preference relation $\boxed{\succsim \subseteq \underline{X} \times \underline{X}}$

$\underline{X} = \{(1,0), (0,1), (1,1)\} \subseteq \mathbb{R}^2$



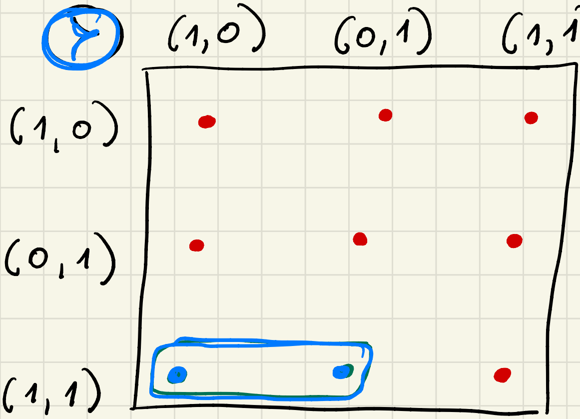


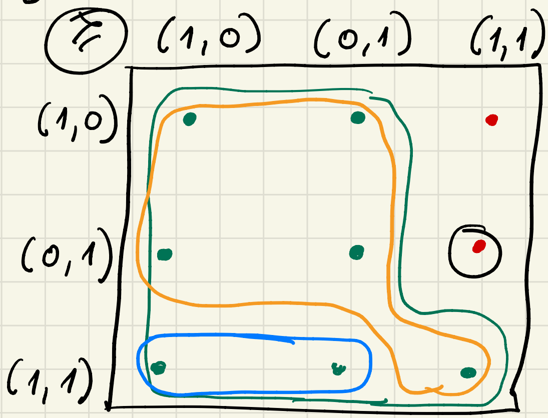
Strict preference relation

$\boxed{\succ \subseteq X \times X}$

$$\boxed{x \succ y \iff x \succsim y \text{ and } y \not\succsim x}$$

"x is strictly better than y".



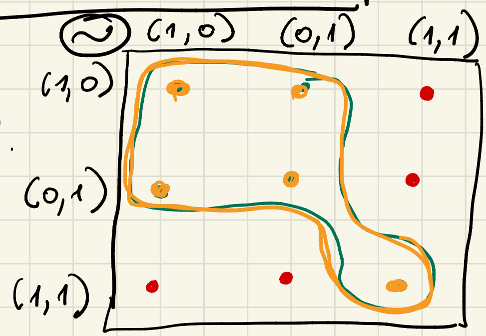


INDIFFERENCE

$\boxed{\sim \subseteq X \times X}$

$$\boxed{x \sim y \iff x \succsim y \text{ and } y \succsim x}$$

"x is equally good as y".



Question A.1:

(a) $\succ \subseteq \succsim$

(b) $\sim \subseteq \succsim$

(c) $\succ \cup \sim = \succsim$

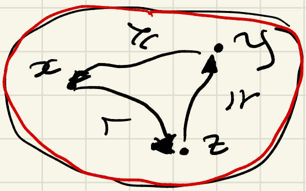
AXIOMS : (seemingly) natural and innocent assumptions on $\succsim$.

(A1) Completeness : for any pair $x, y \in X$

$$\boxed{x \succsim y \quad \text{or} \quad y \succsim x}$$

(A2) Transitivity : for any $x, y, z \in X$

$$\boxed{\text{if } x \succsim y \text{ and } y \succsim z, \text{ then } x \succsim z}$$



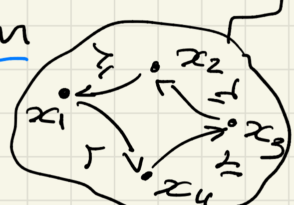
TRANSITIVITY RULES OUT SUCH CASES.

Without transitivity the DM can be exploited as a money pump.

Lemma : If $\succsim$ satisfy (A1) and (A2), then there is no cycle of the form

$$\boxed{x_1 \succsim x_2 \succsim x_3 \succsim x_4 \succ x_1}$$

In fact the same argument I can make for cycles of any length.



If $x_1 \succsim x_2$, $x_2 \succsim x_3$, $x_3 \succsim x_4$, then $x_4 \not\succ x_1$ (or $x_1 \succsim x_4$)

by (A2) I obtain $x_1 \succsim x_3$.

by (A2) I obtain $\boxed{x_1 \succsim x_4}$ QED

For any $x_1, x_2, \dots, x_n$ (where n is an arbitrary finite number)

if $x_1 \succsim x_2$, $x_2 \succsim x_3, \dots, x_{n-1} \succsim x_n$, it cannot be the case that $x_n \succ x_1$ (it is necessarily that $x_1 \succsim x_n$).

Proof by induction:

① Prove the result for $n=3$

② If we assume that the result holds for $n-1$, then show that it holds for n

Assume that there are no cycles of length $n-1$. This means that: if $x_1 \succsim x_2 \succsim \dots \succsim x_{n-1}$ then $x_1 \succsim x_{n-1}$.

Then I want to prove that there is no cycle of length n, viz. if $x_1 \succsim x_2 \succsim \dots \succsim x_{n-1} \succsim x_n$ then $x_1 \succsim x_n$

$x_{n-1} \succsim x_n$

by the previous step, $x_1 \succsim x_{n-1}$

by (A_2), $\boxed{x_1 \succsim x_n}$ QED

Question A.2: if X is finite and $\succsim$ satisfies (A1) and (A2), then there is a most preferred outcome (and a least preferred outcome)

There is some $x \in X$ such that $x \succsim y$ for all $y \in X$

$\Updownarrow$

There is some $x \in X$ such that for no $y \in X$ is it the case that $y \succ x$.

Direct proof: X contains n elements.

Pick randomly some $x_1 \in X$. There are two possibilities:

① $\forall x \in X : x_1 \succsim x$ (I am done)

② $\exists x_2 \in X : x_2 \succ x_1$

Take x_2 and again there are two possibilities

① $\forall x \in X : x_2 \succsim x$ (I am done)

② $\exists x_3 \in X : x_3 \succ x_2$

Q: Can x_3 be the same as x_1?

If yes (i.e, if $x_1 = x_3$) then $x_1 \succ x_2$ and $x_2 \succ x_1$.

But this cannot happen if the preferences are transitive ((A1) in Problem A.1).

$x_3 \succ x_2 \succ x_1$

By induction, we can show that for each step $k \leq n$, I will have $x_k \succ x_{k-1} \succ \cdots \succ x_2 \succ x_1$

Continuing to n, we obtain

$$x_n \succ x_{n-1} \succ \cdots \succ x_2 \succ x_1$$

There are two possibilities:

1. $\forall x \in \underline{X} : x_n \succsim x$ (I am done)
2. $\exists x \in \underline{X} : x \succ x_n$, then it must be that

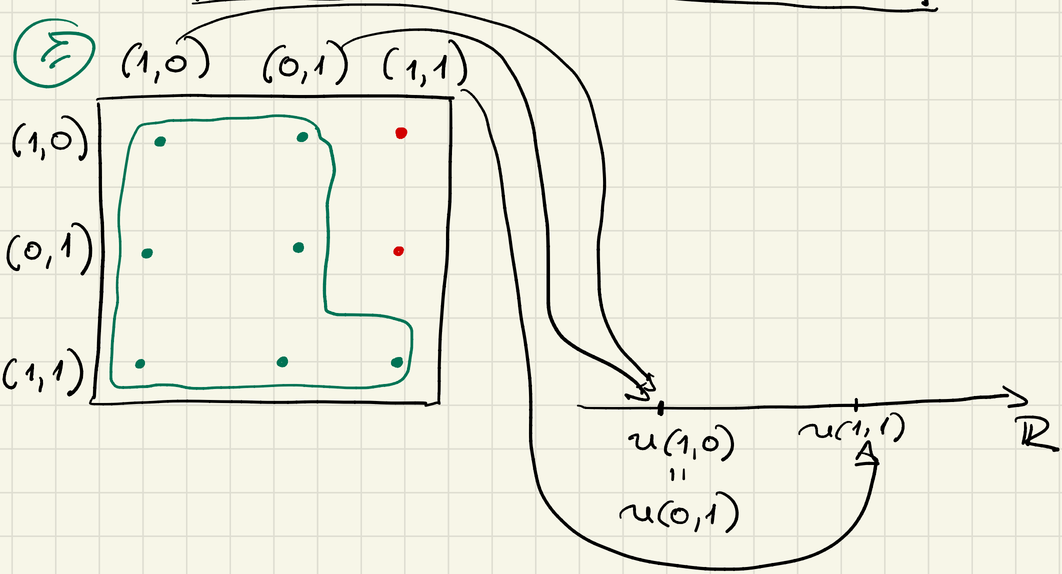
$$\boxed{x} \succ x_n \succ x_{n-1} \succ \cdots \succ x_2 \succ x_1$$

This means that there exists a cycle. However as we have shown, in the presence of transitivity, there are no cycles.

QED.

UTILITY REPRESENTATION: a function $u: X \to \mathbb{R}$ such that for any $x, y \in X$

$$\boxed{x \succsim y \iff u(x) \geq u(y)}$$



Q: If u is a utility representation of $\succsim$, is it unique?

Question A.3: If u represents $\succsim$, then v (which is defined by $v(x)=f(u(x))$ for some strictly increasing $f:\mathbb{R}\to\mathbb{R}$) also represents $\succsim$.

For any $x, y \in X$:

$$\boxed{x \succsim y} \iff u(x) \geq u(y)$$

$$\iff f(u(x)) \geq f(u(y))$$

$$\iff \boxed{v(x) \geq v(y)}$$

Hence v is a utility representation of $\succsim$.

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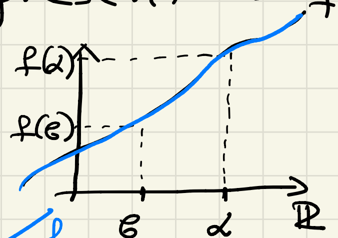
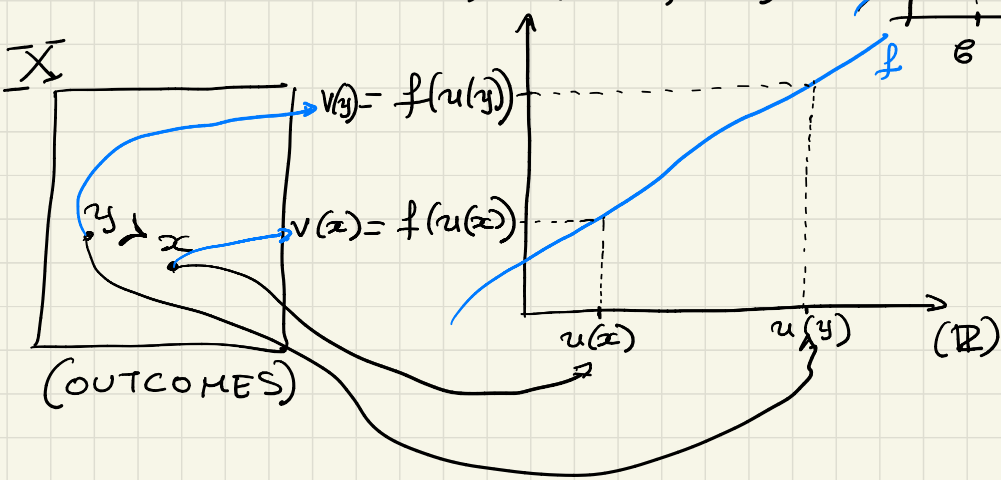
f is strictly increasing when

$$\alpha \geq \beta \iff f(\alpha) \geq f(\beta)$$





PROPERTIES OF UTILITY: mathematical properties of the function $u: X \to \mathbb{R}$

Examples: increasing, differentiable, convex, continuous, etc

Property is relevant if we use it in order to make predictions, or to classify the DM into some category, or in general to say something about the DM's behavior.

ORDINAL PROPERTIES: all utility representations of $\succsim$ have this same property.

CARDINAL PROPERTIES: only some (but not all) of the utility representations of $\succsim$ satisfy this property.

A utility function is ordinal if we only look at ordinal properties, while it is cardinal if we also look at cardinal properties.

Question A.4: Both $u(x) = \ln(1+x)$ and $v(x) = x^2$ represent the same preferences over $X = \mathbb{R}_+$. Relate to A.3. Is "the utility function is increasing" ordinal or cardinal? How about "the utility function is convex"?

For any two $x, y \in X = \mathbb{R}_+$

$$x \succsim y \overset{u \text{ repres.} \succsim}{\iff} u(x) \geq u(y) \iff \ln(1+x) \geq \ln(1+y)$$

$$\iff 1+x \geq 1+y \iff x \geq y \iff x^2 \geq y^2$$

$$\overset{v \text{ repres.} \succsim}{\iff} v(x) \geq v(y)$$

u represents $\succsim \iff v = \underbrace{f \circ u}$ also represents $\succsim$ for some strictly increasing $f: \mathbb{R} \to \mathbb{R}$.

composition of f and u, and it is defined by $f(u(x))$

I will now show that $v(x) = f(u(x))$ for some strictly increasing f.

$$\boxed{v(x) = f(u(x))} \iff \underbrace{x^2}_{(*)} = f(\underbrace{\ln(1+x)}_{\alpha}) \quad (*)$$

Change of variable: $\ln(1+x) = \alpha \iff$

$$\iff e^{\ln(1+x)} = e^{\alpha} \iff 1+x = e^{\alpha}$$

$$\iff x = e^{\alpha} - 1 \iff x^2 = (e^{\alpha} - 1)^2$$

$$\iff \underbrace{x^2}_{(*)} = e^{2\alpha} - 2e^{\alpha} + 1$$

$$(*) \Rightarrow \boxed{e^{2\alpha} - 2e^{\alpha} + 1 = f(\alpha)}$$

$$f'(\alpha) = 2e^{2\alpha} - 2e^{\alpha} = 2(e^{2\alpha} - e^{\alpha}) > 0$$

$\Rightarrow f$ is strictly increasing

Hence by A.3: if u is a utility representation of $\succsim$, so is v.

I have already established that $x \succsim y \iff x \geq y \iff \hat{u}(x) \geq \hat{u}(y)$ for any strictly increasing $\hat{u}: X \to \mathbb{R}$.

"utility being strictly increasing wrt money" is an ordinal property

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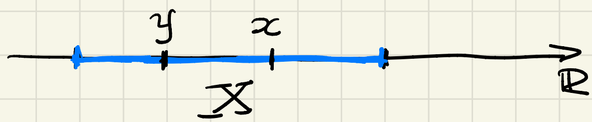
"utility being convex" is cardinal (u is concave / v convex)

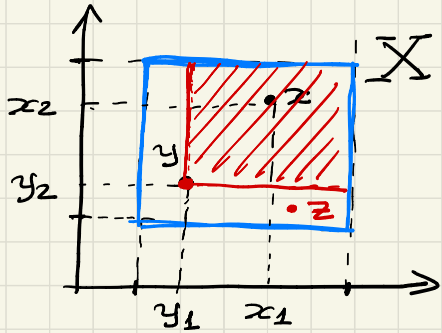
(A_0) MONOTONICITY: for any $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ such that $x_k \geq y_k$ for all $k=1,\dots,n$, then $x \succsim y$.

EXAMPLES: • $X \subseteq \mathbb{R}$ monetary payoffs



$x \geq y \Rightarrow x \succsim y$

• $X \subseteq \mathbb{R}^2$ bundles of two goods



$x_1 \geq y_1$ and $x_2 \geq y_2$

$\Longrightarrow x \succsim y$

(Remark: this not "if and only if")

• $X \subseteq \mathbb{R}^4$ allocations of monetary payoffs to four individuals:

$x_1 \geq y_1,\ x_2 \geq y_2,\ x_3 \geq y_3,\ x_4 \geq y_4 \Longrightarrow$

$x \succsim y$

Question A.5: u represents $\succsim$. Then:

$$\boxed{\succsim \text{ satisfies } (A_0) \iff u \text{ is increasing}}$$

$X \subseteq \mathbb{R}^n$, u is increasing if

Def: for any $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ such that $x_k \geq y_k$ for all $k = 1, \dots, n$ it will be the case that $u(x) \geq u(y)$

(Remark: if for instance u is differentiable, then all partial derivatives are positive).

$$\succsim \text{ satisfies } (A_0) \overset{\text{def. of } (A_0)}{\iff} \text{for all } x, y \text{ with } x_k \geq y_k \text{ for all } k,\ x \succsim y$$

$$\overset{u \text{ is a utility repr.}}{\iff} \text{for all } x, y \text{ with } x_k \geq y_k \text{ for all } k,\ u(x) \geq u(y)$$

$$\overset{\text{def of increasing function}}{\iff} u \text{ is increasing}$$

QED

Question A.6: A property (P) is satisfied by a utility function u. Then:

$$\boxed{P \text{ is ordinal} \iff \text{it is satisfied by every } v = f \circ u \text{ for strictly increasing } f.}$$

Recall $v = f \circ u$ means $v(x) = f(u(x))$

Recall (from Question A.3)

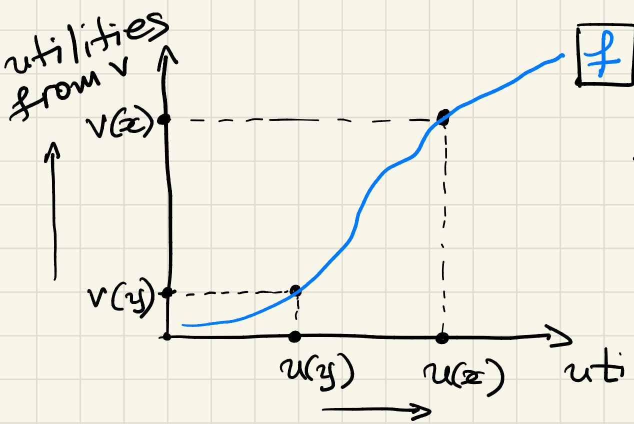
$$v = f \circ u \text{ for strictly increasing } f$$

$$\Updownarrow \; \boxed{?}$$

$$v \text{ is a utility representation}$$

Begin with u, v representations of $\succsim$.

$$u(x) \geq u(y) \iff x \succsim y \iff v(x) \geq v(y)$$



it will be the case that $v(x) = f(u(x))$.

Question A.7 : $u(x) = 20{,}000x - x^2$.

Does u represent some preferences satisfying (A1) - (A2)? How about (A0)? Draw the graph of u, and describe. Ordinal vs. cardinal preferences. New utility representation with different cardinal preferences.

$$\boxed{x \succsim y \iff u(x) \geq u(y)}$$

(A1) Take any $x, y \in X = \mathbb{R}_+$

$u(x) \geq u(y)$ or $u(x) \leq u(y) \implies$

$x \succsim y$ or $x \precsim y$

(A2) Take any $x, y, z \in X$ such that $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Then, $u(x) \geq u(z)$

$\implies x \succsim y$ and $x \succsim z$ imply $x \succsim z$.

(A0) $x = 10{,}000 = 10^4$ $\implies u(10{,}000) = 20{,}000 \cdot 10^4 - 10^8$

$y = 10{,}000{,}000 = 10^7$

$= 2 \cdot 10^8 - 10^8$

$= 10^8 > 0$

$u(10{,}000) > u(10{,}000{,}000)$

even though $10{,}000 < 10{,}000{,}000$

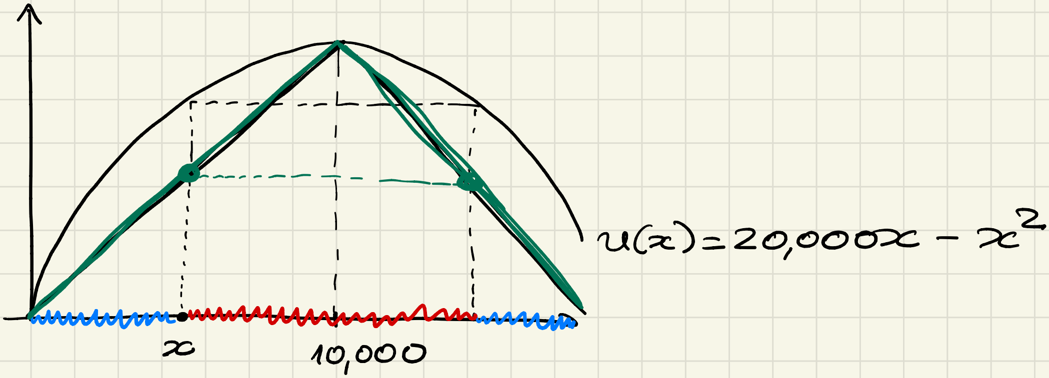
$u(10{,}000{,}000) = 20{,}000 \cdot 10^7 - 10^{14}$

$= 2 \cdot 10^{11} - 10^{14} < 0$

Hence monotonicity is violated

$$u'(x) = 20{,}000 - 2x = 0$$

$$\iff x = 10{,}000$$



Properties of u:

ORDINAL
- it is strictly increasing in $[0, 10{,}000]$ and strictly decreasing in $[10{,}000, \infty)$

CARDINAL
- it is strictly concave

Theorem A.1: If X is finite, and $\succsim$ satisfy (A1)-(A2), then there is a utility function.

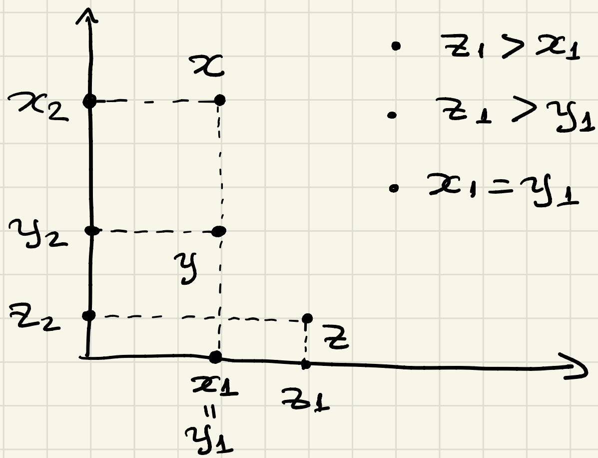
Recall that if X is finite and $\succsim$ satisfy (A1)-(A2), then there exists a most/least preferred outcome (Question A.2).

By finiteness of X, we take it that there are n elements, and by Q.A2, take the least preferred one, call it x_1 and assign to it utility $u(x_1)=0$. Then I am left with a new set $X_1 := X \setminus \{x_1\}$. Since X_1 is finite it has a least preferred outcome say x_2. There are two possibilities. Either $x_1 \sim x_2$ in which case I assign $u(x_2) = u(x_1) = 0$, or $x_1 \prec x_2$ in which case $u(x_2) = u(x_1) + 1$. I continue inductively, at each step defining $X_k = X_{k-1} \setminus \{x_k\}$ and choosing from X_k the least preferred outcome x_{k+1}. There will be two cases, either $x_{k+1} \sim x_k$ in which case $u(x_{k+1}) = u(x_k)$ or $x_{k+1} \succ x_k$ in which case $u(x_{k+1}) = u(x_k) + 1$. This will eventually stop, and I will have assigned utilities to all outcomes ($X_n = \emptyset$).

LEXICOGRAPHIC PREFERENCES : $X = \mathbb{R}^2_+$

$x = (x_1, x_2)$ $\quad$ $y = (y_1, y_2)$

$$\boxed{x \succ y \iff \text{either } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \geq y_2}$$



- $z_1 > x_1 \Rightarrow z \succ x$
- $z_1 > y_1 \Rightarrow z \succ y$
- $x_1 = y_1$ and $x_2 > y_2 \Rightarrow x \succ y$

Question A.9: $\succsim$ satisfy (A0) - (A1) - (A2).

(A1) $x = (x_1, x_2)$, $y = (y_1, y_2)$

There are three cases:

- $x_1 > y_1 \Rightarrow x \succ y$
- $x_1 < y_1 \Rightarrow y \succ x$
- $x_1 = y_1 \Rightarrow$
  - $x_2 > y_2 \Rightarrow x \succ y$
  - $x_2 < y_2 \Rightarrow y \succ x$
  - $x_2 = y_2 \Rightarrow x \sim y$

$\Big\}$ $x \succsim y$ or $y \succsim x$.

(A2) $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, such that $x \succsim y$ and $y \succsim z$

Take the following cases:

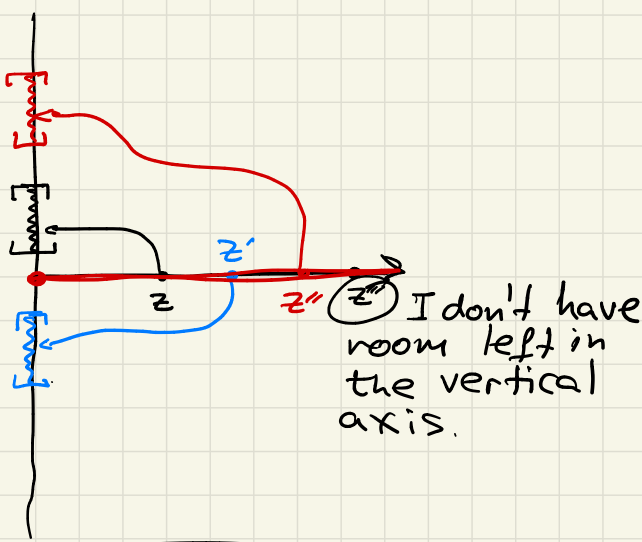
- $x_1 > y_1$
  - $\rightarrow y_1 > z_1 \Rightarrow x_1 > z_1 \rightarrow \boxed{x \succsim z}$
  - $\rightarrow y_1 = z_1$ and $y_2 \geq z_2 \Rightarrow x_1 > z_1 \Rightarrow \boxed{x \succsim z}$
- $x_1 = y_1$ and $x_2 \geq y_2$
  - $\rightarrow y_1 > z_1 \Rightarrow x_1 > z_1 \Rightarrow \boxed{x \succsim z}$
  - $\rightarrow y_1 = z_1$ and $y_2 \geq z_2 \Rightarrow$ $x_1 = z_1$ and $x_2 \geq z_2 \Rightarrow \boxed{x \succsim z}$

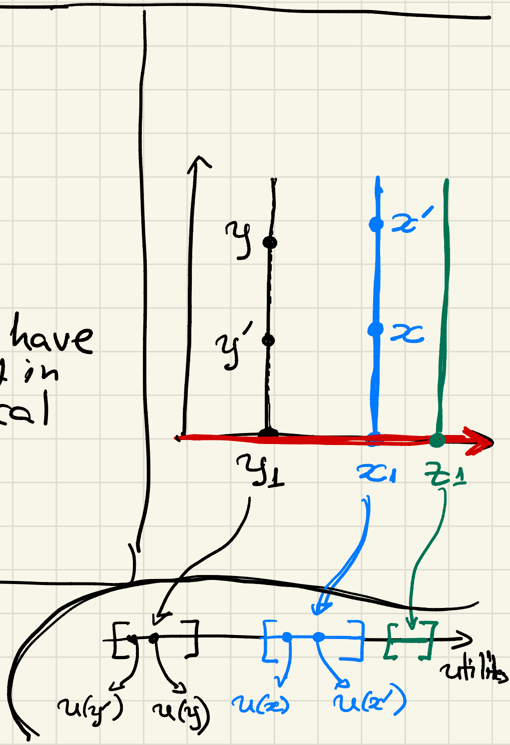
(A0) $x=(x_1,x_2)$ $y=(y_1,y_2)$ such that $x_1 \geq y_1$ and $x_2 \geq y_2$. Then, I can rewrite this as two cases:

$$\left.\begin{array}{l} \bullet\ x_1 > y_1 \text{ and } x_2 \geq y_2 \\ \bullet\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{array}\right\} \Rightarrow x \succsim y.$$

Question A.10: $\succsim$ does not have a utility representation.

Hint: You cannot fit uncountably many intervals in $\mathbb{R}$. Formally if every $z \in \mathbb{R}_+$ is associated with some interval $[a_z, b_z]$, it is not possible that $[a_z, b_z] \cap [a_{z'}, b_{z'}] = \emptyset$ for each z, z'.





CHOICE PROBLEM (DECISION PROBLEM)

Set of actions / choices and each of these choices yields an outcome from X.

In this sense, we write $\boxed{A} \subseteq X$

By $\mathcal{A}$, I denote the set of all choice problem's.

RATIONAL CHOICE (defined relatively to the choice problem A and preferences $\succsim$)

some $a \in A$ such that $a \succsim b$ for all $b \in A$.

In case $\succsim$ are represented by u, then a rational choice is a utility maximizer. That is, some $a \in A$ such that $u(a) \geq u(b)$ for any $b \in A$.

Question B.1: if X is finite and $\succsim$ satisfy (A1)-(A2), then there is a rational choice.

By Q. A.2, $A \subseteq X$ is also finite, and therefore there is a most preferred outcome in A, which is a rational choice.

Question B.2: $u(x) = x^2$ represents $\succsim$ over $\mathbb{R}_+$

Some A that does not have a rational choice.

$A = [0,1) \subseteq \mathbb{R}_+$: u does not have a maximizer in A

EXPERIMENT: contains observations from a sequence of choice problems, $\boxed{\mathcal{E} \subseteq \mathcal{A}}$. Two cases

- complete experiment: $\mathcal{E} = \mathcal{A}$
- incomplete experiment: $\mathcal{E} \subsetneq \mathcal{A}$

DATASET: observed choices in each choice problem in the experiment

- $\boxed{\text{Choice correspondence}: C(A) \subseteq A \text{ for each } A \in \mathcal{E}.}$
- $\boxed{\text{Choice function}: c(A) \in A \text{ for each } A \in \mathcal{E}}$

RATIONALIZING CHOICE:

> "Is there a preference relation $\succsim$ satisfying the axioms that I want to test, which is consistent with the data I have observed"?

- If NO, then I reject the axioms
- If YES, then I cannot reject the axioms and I take it that the DM acts <u>as if</u> he was driven by such axioms.

METHODOLOGY: Revealed Preference.

Example : $X = \{$Vegeterian (a), Pasta (b), Steak (c), Sushi (d), Tacos (e)$\}$

Restaurant 1 : $A = \{a, b, c, d\}$

Restaurant 2 : $B = \{b, d, e\}$

$\Big\}\ \boxed{\mathcal{E}}$ (incomplete experiment)

$C = \{a, c, e\}$, $C \notin \mathcal{E}$

CHOICE CORRESPONDENCE : $\forall A \in \mathcal{E}$, I observe $\boxed{C(A) \subseteq A}$, all choices that the DM makes from this menu.

In the previous example $C(A) = \{a, c\} \subseteq A$

$C(B) = \{d, e\} \subseteq B$

RATIONALIZING A CHOICE CORRESPONDENCE

We say that C is rationalized by some preference relation $\succsim$ (which satisfies certain axioms), if for each $A \in \mathcal{E}$:

$$\boxed{C(A) = \{a \in A : a \succsim b \text{ for all } b \in A\}} \quad (*)$$

In other words, the behavior observed in C is consistent with the axioms that I am testing if I can find some $\succsim$ satisfying these axioms such that $(*)$ holds.

This preference relation is not necessarily unique (in fact, almost always it is not). I cannot identify exactly the preferences.

DIRECT REVEALED PREFERENCE: a choice $a \in X$ is directly revealed preferred to choice $b \in X$ (and we write $\boxed{a \succsim_c b}$) whenever

there is some $A \in \mathcal{E}$ containing both a, b such that $\boxed{a \in C(A)}$.

DIRECT STRICT REVEALED PREFERENCE: a choice $a \in X$ is strictly directly revealed preferred to $b \in X$ (and we write $\boxed{a \succ_c b}$) whenever there is some $A \in \mathcal{E}$ containing both a, b such that $\boxed{a \in C(A) \text{ and } b \notin C(A)}$

Question B.3: Example where $a \succsim_c b$ and $b \succsim_c a$

$X = \{a, b, c\}$ $\qquad A = \{a, b\}$ $\qquad C(A) = \{a\}$

Vegeterian Pasta Steak $\qquad B = \{a, b, c\}$ $\qquad C(B) = \{b\}$

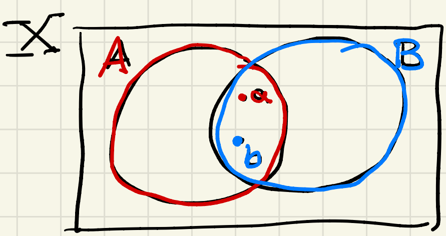
From $C(A)$ we obtain $a \succ_c b$

From $C(B)$ we abtain $b \succ_c a$

WEAK AXIOM OF REVEALED PREFERENCE (WARP)

For any $A, B \in \mathcal{E}$ with $a, b \in A \cap B$,

$$\boxed{a \in C(A) \text{ and } b \in C(B) \Rightarrow b \in C(A) \text{ and } a \in C(B)}$$



Question B.4: WARP violated

There exist $A, B \in \mathcal{E}$ with $a, b \in A \cap B$

such that:

although $a \in C(A)$ and $b \in C(B)$, it will be $a \notin C(B)$ or $b \notin C(A)$.

Theorem B.1: Fix experiment $\mathcal{E}$:

(i) If C violates WARP, then C cannot be rationalized.

(ii) If C satisfies WARP in complete $\mathcal{E}$, then C can be rationalized

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Question B.5: $X = \{a, b, c\}$

$\mathcal{E} = \{\{a,b\}, \{b,c\}, \{a,c\}\}$

$C(\{a,b\}) = \{a\}$
$C(\{b,c\}) = \{b\}$
$C(\{a,c\}) = \{c\}$

WARP is satisfied.

Every pair $A, B \in \mathcal{E}$ such that $a, b \in A \cap B$

There are no such pairs of choice problems

WARP is trivially satisfied.

$C(A) = \{a \in A : a \succsim b \text{ for all } b \in A\}$

$$\left.\begin{array}{l} C(\{a,b\}) = \{a\} \Rightarrow a \succ b \\ C(\{b,c\}) = \{b\} \Rightarrow b \succ c \\ C(\{a,c\}) = \{c\} \Rightarrow c \succ a \end{array}\right\} \text{Transitivity is violated.}$$

Question B.6: $C(\{a,b,c\})$

$$\left.\begin{array}{l} \text{Suppose } \boxed{a \in C(\{a,b,c\})} \\ c \in C(\{a,c\}) \\ a, c \in \{a,c\} \cap \{a,b,c\} \end{array}\right\} \Rightarrow \begin{array}{l} c \in C(\{a,b,c\}) \\ \boxed{a \in C(\{a,c\})} \end{array}$$

must be false

contradiction.

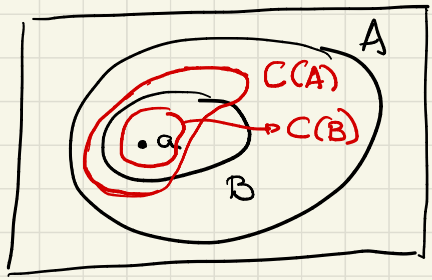
Question B.7: If C satisfies WARP then for all $A, B \in \mathcal{E}$ with $B \subseteq A$:

(i) PROPERTY α: $a \in B$ and $a \in C(A) \Rightarrow a \in C(B)$

(ii) PROPERTY β: $a, b \in C(B)$ and $a \in C(A) \Rightarrow b \in C(A)$

Moreover, if $\mathcal{E}$ is complete then the converse holds, ie, properties α and β imply WARP

$a \in B \xRightarrow{B \subseteq A} a \in A$



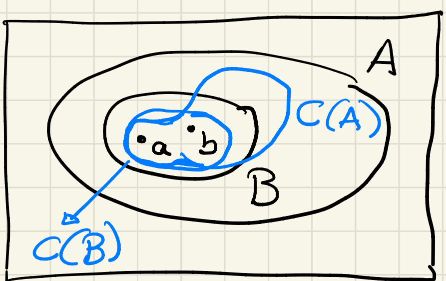
case 1: there is another $b \in C(B) \xRightarrow{C(B) \subseteq B} b \in B \xRightarrow{B \subseteq A} b \in A$

$$\left.\begin{array}{l} \text{Hence, } a, b \in A \cap B \\ \text{Plus, } b \in C(B) \\ \qquad a \in C(A) \end{array}\right\} \xRightarrow{\text{WARP}} \begin{array}{l} \boxed{a \in C(B)} \checkmark \\ b \in C(A) \end{array}$$

case 2: there is no other $b \in C(B)$

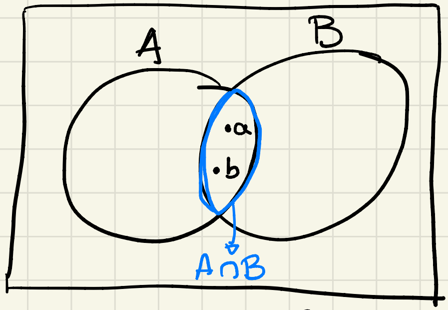
Since $C(B) \neq \emptyset$, I obtain $\boxed{a \in C(B)}$ ✓

$a, b \in C(B) \xRightarrow{C(B) \subseteq B} a, b \in B \xRightarrow{B \subseteq A} a, b \in A$



$$\left.\begin{array}{l} \text{Hence, } a, b \in A \cap B \\ \qquad b \in C(B) \\ \qquad a \in C(A) \end{array}\right\} \xRightarrow{\text{WARP}} \begin{array}{l} a \in C(B) \\ \boxed{b \in C(A)} \checkmark \end{array}$$

Take $A, B \in \mathcal{E}$, such that



① $a, b \in A \cap B$

② $a \in C(A)$

③ $b \in C(B)$

Since $\mathcal{E}$ is complete, $A \cap B \in \mathcal{E}$ $(A \cap B \subseteq A)$
$(A \cap B \subseteq B)$

$$① + ② \xrightarrow{\alpha} a \in C(A \cap B) \quad ③$$

$$① + ② \xrightarrow{\alpha} b \in C(A \cap B) \quad ④$$

$$③ + ④ \Longrightarrow \left.\begin{array}{l} a, b \in C(A \cap B) \\ a \in C(A) \end{array}\right\} \xrightarrow{\beta} \boxed{b \in C(A)}$$

$$③ + ④ \Longrightarrow \left.\begin{array}{l} a, b \in C(A \cap B) \\ b \in C(B) \end{array}\right\} \xrightarrow{\beta} \boxed{a \in C(B)}$$

Question B.8 : (i) If C violates α or β then C cannot be rationalized.

(ii) If C satisfies α and β in a complete $\mathcal{E}$, then C can be rationalized.

Thm B.1 : (i) $\boxed{\text{WARP violated}} \Rightarrow \boxed{C \text{ not rationalized}}$

(ii) $\boxed{\text{WARP satisfied in complete } \mathcal{E}} \Rightarrow \boxed{C \text{ rationalized}}$

Question B.7 : (i) $\boxed{\text{WARP satisfied}} \Rightarrow \boxed{\alpha, \beta \text{ satisfied}}$

(ii) $\boxed{\alpha \text{ and } \beta \text{ satisfied in complete } \mathcal{E}} \Rightarrow \boxed{\text{WARP satisfied}}$

(i) C violates α or $\beta \xRightarrow{B.7.i}$ WARP violated

$\xRightarrow{B.1.i}$ C not rationalized

(ii) α, β satisfied in complete $\mathcal{E} \xRightarrow{B.7.ii}$ WARP satisfied (in complete $\mathcal{E}$)

$\xRightarrow{B.1.ii}$ C rationalized

$p \longrightarrow q$

$\neg q \longrightarrow \neg p$

(not q) (not p)

INDIRECT REVEALED PREFERENCE:

Take C, and I say that a is <u>indirectly revealed preferred</u> to b whenever:

$$\boxed{a \succ_C a_1 \succ_C a_2 \succ_C \cdots \succ_C a_N \succ_C b}$$

In this case I write $a \succ_C^* b$

GENERALIZED AXIOM OF REVEALED PREFERENCE (GARP): For any $a, b \in X$, it cannot be

$$\boxed{\underbrace{a \succ_C^* b}\quad \text{and} \quad b \succ_C a}$$

If you replace here $a \succ_C^*$ with $a \succ_C b$ then instead of GARP you get WARP (Problem B.1).

Theorem B.2: For any $\mathcal{E}$ and C:

(i) C violates GARP $\Longrightarrow$ C not rationalized

(ii) C satisfies GARP $\Longrightarrow$ C is rationalized

Question B.9: $\mathcal{E} = \{\{a,b\},\{b,c\},\{a,c\}\}$

$$\underbrace{C(\{a,b\}) = \{a\}}_{a \succ_C b}, \quad \underbrace{C(\{b,c\}) = \{b\}}_{b \succ_C c}, \quad \underbrace{C(\{a,c\}) = \{c\}}_{c \succ_C a}$$

$$\underbrace{a \succ_C b \quad\quad b \succ_C c}_{a \succ_C b \succ_C c \Longrightarrow}$$

$$\boxed{\begin{array}{c} c \succ_C a \\ \\ a \succ_C^* c \end{array}}$$

GARP violated

CHOICE FUNCTION :

Earlier when we focused on choice correspondences, we essentially assumed that the experimenter can observe <u>all choices</u> made in each choice problem.

Now we will instead assume that the experimenter can observe only <u>one choice</u>

For each $A \in \mathcal{E}$, $c(A) \in A$ is the observed choice, and c is called the choice function.

<u>Rationilizing</u>: a choice function c is rationalized by some $\succsim$, if for all $A \in \mathcal{E}$:

$$\boxed{c(A) \in \{a \in A : a \succsim b \text{ for all } b \in A\}}$$

$(c(A) \succsim b \quad \text{for all } b \in A)$

<u>Question B.10: every choice function is rationalized by a complete and transitive $\succsim$.</u>

Take $\succsim$ such that $a \sim b$ for all $a, b \in X$

This means that for every $A \in \mathcal{E}$,

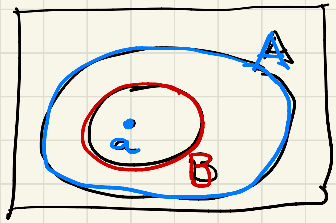
$c(A) \in \{a \in A : a \succsim b \text{ for all } b \in A\} = A$

Hence $\succsim$ rationalizes c.

(NI) <u>NO INDIFFERENCE</u>: for any $x, y \in X$, either $x \succ y$ or $y \succ x$.

<u>PROPERTY α</u>: For all $A, B \in \mathcal{E}$ with $B \subseteq A$:

$$c(A) \in B \Rightarrow c(A) = c(B)$$



<u>Question B.11</u>: Relate property α for choice correspondences to property α for choice functions.

Property α (for correspondences): if $B \subseteq A$

$$a \in B \text{ and } a \in C(A) \Rightarrow a \in C(B)$$

<u>CORR</u>: $a \in B$ and $a \in \boxed{C(A)}$ $\Rightarrow$ $a \in C(B)$

<u>FUNCT</u>: $\boxed{c(A)} \in B$ $\Rightarrow$ $c(A) = c(B)$

the same assuming that $c(A) = C(A)$

the same assuming that $c(A) = C(A)$ and $c(B) = C(B)$

Theorem B.3: For $\mathcal{E}$ and c:

(i) c violates Property α $\Rightarrow$ c cannot be rationalized by any $\succsim$ satisfying completeness, transitivity and no-indifference.

(ii) c satisfies Property α in a complete $\mathcal{E}$ $\Rightarrow$ c can be rationalized by some $\succsim$ satisfying completeness, transitivity, no-indifference

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Question B.12: $X = \{a,b,c\}$, $\mathcal{E} = \{\{a,b\},\{b,c\},\{a,c\}\}$

$c(\{a,b\}) = \{a\}$ $\quad$ $c(\{b,c\}) = \{b\}$ $\quad$ $c(\{a,c\}) = \{c\}$

Prove that c satisfies Property α. Moreover it cannot be rationalized.

Trivial: Check for all $A, B \in \mathcal{E}$ such that $B \subseteq A$. Well, here there are no such choice problems.

In B.5 I have shown that there is no complete and transitive preference relation that would have given me this data.

DECISION THEORY UNDER UNCERTAINTY

Question C.1:

OBJECTIVE UNCERTAINTY: Probabilities that are objectively known to everyone

SUBJECTIVE UNCERTAINTY: Subjective probabilities which may differ across individual

OBJECTIVE UNCERTAINTY / RISK:

Lottery: A random experiment yielding some monetary outcomes together with the respective probabilities.

$$\boxed{P = \Big(\underbrace{P(x_1)}_{\text{prob of } x_1}\times \underbrace{\boxed{x_1}}_{\in\mathbb{R}}, \underbrace{P(x_2)}_{\text{prob of } x_2}\times \underbrace{\boxed{x_2}}_{\in\mathbb{R}}, \dots, \underbrace{P(x_n)}_{\text{prob of } x_n}\times \underbrace{\boxed{x_n}}_{\in\mathbb{R}}\Big)}$$

- $P(x_1) + P(x_2) + \dots + P(x_n) = 1$
- Finitely many outcomes $x_1, x_2, \dots, x_n$

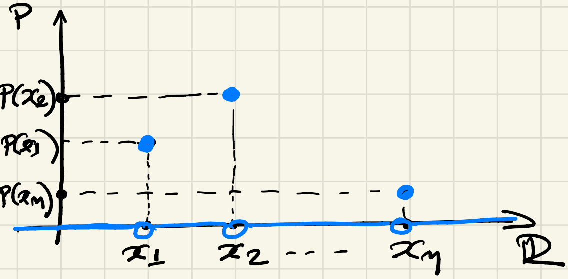
Set of all lotteries is denoted by $\mathcal{L}$.

$(1\times x) = x$: each $x \in X$ is identified by lottery $(1\times x)$.

In this sense I write $X \subseteq \mathcal{L}$

Question C.2: Only the probabilities matter not the actual experiment. Provide an example

$$P = (P(x_1) \times x_1, \dots, P(x_m) \times x_m)$$



Example: Take $\boxed{P = \left(\frac{1}{2} \times 0, \frac{1}{2} \times 20\right)}$

- Toss a coin and:
  - If "H" comes, then pay 0
  - If "T" comes, then pay 20
- Roll a die and:
  - If "⚀, ⚁, ⚂", then pay 0
  - If "⚃, ⚄, ⚅", then pay 20

PREFERENCES over $\mathcal{L}$, described by $\succsim$

"$p \succsim q$" means lottery p is at least as good as lottery q.

Strict preference and indifference are defined in exactly the same way

Preferences induced over X: Whenever I have $\succsim$ over $\mathcal{L}$, I also have preferences over degerate lotteries, and this gives me preferences over X:

$$\boxed{(1 \times x) \succsim (1 \times y) \Longleftrightarrow x \succsim y}$$

Induced preferences over X are monotonic (i.e., $x \geq y \Rightarrow (1 \times x) \succsim (1 \times y) \Rightarrow x \succsim y$).
It will not be the case that $x \sim y$ for all $x, y \in \mathbb{R}$.

In principle, I will be interested in $\succsim$ over $\mathcal{L}$ that satisfy completeness and transitivity.

Question C.3: $p = (p(x_1) \times x_1, \dots, p(x_m) \times x_m)$

$q = (q(y_1) \times y_1, \dots, p(y_L) \times y_2)$:

$$\boxed{p \succsim q \iff \sum_{k=1}^{K} p(x_k) \cdot x_k \geq \sum_{l=1}^{L} q(y_l) \cdot y_l}$$

$$\sum_{k=1}^{K} p(x_k) p_x \geq \sum_{l=1}^{L} q(y_l) \cdot y_l \ \underline{\underline{\text{OR}}} \ \sum_{k=1}^{K} p(x_k) \cdot x_k \leq \sum_{l=1}^{L} q(y_l) \cdot y_l$$

$$\iff \quad p \succsim q \quad \underline{\underline{\text{OR}}} \quad p \precsim q$$

complete holds.

$p \succsim q$ and $q \succsim r \implies$

$$\sum_{k=1}^{K} p(x_k) \cdot x_k \geq \sum_{l=1}^{L} q(y_l) \cdot y_l \geq \sum_{m=1}^{M} r(z_m) \cdot z_m$$

$\implies p \succsim r$

transitivity holds

Take $x \geq y \implies \underbrace{1 \cdot x}_{\substack{\text{expected} \\ \text{payoff of} \\ (1 \times x)}} \geq \underbrace{1 \cdot y}_{\substack{\text{expected} \\ \text{payoff of} \\ (1 \times y)}} \implies x \succsim y$

Monotonicity holds.

Question C.4 :

$$\boxed{p \succsim q \iff \min\{x_1,\dots,x_k\} \geq \min\{y_1,\dots,y_l\}}$$

$$\left.\begin{array}{c} \min\{x_1,\dots,x_k\} \geq \min\{y_1,\dots,y_l\} \\ \text{or} \\ \min\{x_1,\dots,x_k\} \leq \min\{y_1,\dots,y_l\} \end{array}\right\} \Longrightarrow \begin{array}{c} p \succsim q \\ \text{or} \\ p \precsim q \end{array}$$

completeness holds.

$p \succsim q$ and $q \succsim r \Longrightarrow$

$$\min\{x_1,\dots,x_k\} \geq \min\{y_1,\dots,y_l\} \geq \min\{z_1,\dots,z_m\}$$

$p \succsim r$

transitivity holds.

$$x \geq y \Longrightarrow \underbrace{\min\{x\}}_{\text{the number I use to evaluate } (1\times x)} \geq \underbrace{\min\{y\}}_{\text{the number I use to evaluate } (1\times y)} \Longrightarrow x \succsim y$$

Monotonicity holds.

UTILITY REPRESENTATION: $u: \mathcal{L} \to \mathbb{R}$ represents $\succsim$ over $\mathcal{L}$ whenever

$$\boxed{p \succsim q \iff u(p) \geq u(q)}$$

EXPECTED UTILITY: For each outcome $x \in \mathcal{X}$ I take a (Bernoulli) utility $u(x)$. Then, for each lottery $p \in \mathcal{L}$, I take the corresponding expectation:

$$\boxed{\mathbb{E}_p(u) = \sum_{k=1}^{K} p(x_k) \cdot u(x_k)}$$

Then, I say that $\succsim$ over $\mathcal{L}$ has a (vNM) expected utility representation, if

$$\boxed{p \succsim q \iff \mathbb{E}_p(u) \geq \mathbb{E}_q(u)}$$

for some Bernoulli utility function

Question C.5: Do the preferences in C.3 and C.4 have an EU representation?

C.3: $\boxed{p \succsim q} \iff \sum_{k=1}^{K} p(x_k)\cdot \boxed{x_k} \geq \sum_{l=1}^{L} q(y_l)\cdot \boxed{y_l}$

Take $u(x) = x$ $\quad \sum_{k=1}^{K} p(x_k)u(x_k) \geq \sum_{l=1}^{L} q(y_l)u(y_l)$

$$\boxed{\mathbb{E}_p(u) \geq \mathbb{E}_q(u)}$$

C.4: $p \succsim q \iff \min\{x_1,\dots,x_K\} \geq \min\{y_1,\dots,y_L\}$

$x > y$. Then take $(1\times y)$ and $(\frac{1}{2}\times x, \frac{1}{2}\times y)$

And assume that $\succsim$ have an EU representation.

- $\underbrace{\boxed{(\frac{1}{2}\times x, \frac{1}{2}\times y)}}_{p} \sim \underbrace{\boxed{(1\times y)}}_{q}$

$\min\{x,y\} = y = \min\{y\} \Rightarrow \boxed{p \sim q}$

- There exists a Bernoulli utility function $u(x) > u(y)$, and

$$\mathbb{E}_p(u) = \frac{1}{2}u(x) + \frac{1}{2}u(y) > u(y) = \mathbb{E}_q(u)$$

$\Rightarrow \boxed{p \succ q}$

FIRST ORDER STOCHASTIC DOMINANCE

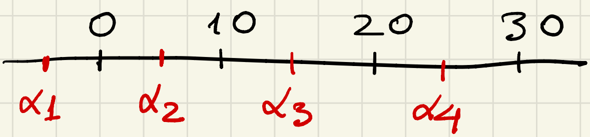
$p = (p(x_1) \times x_1, \dots, p(x_n) \times x_n)$ and $q = (q(y_1) \times y_1, \dots, q(y_L) \times y_L)$

we say that p FOSD q whenever

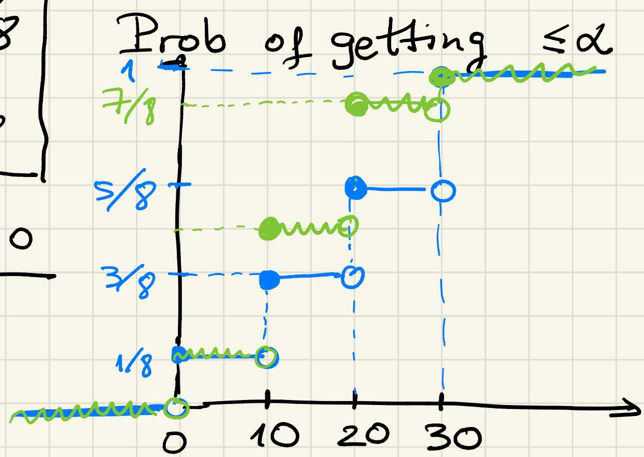
for all $\alpha \in \mathbb{R}$

$$\boxed{\underbrace{\sum_{x_k \geq \alpha} p(x_k)}_{\text{probability of getting at least } \alpha \text{ from } p} \geq \underbrace{\sum_{y_\ell \geq \alpha} q(y_\ell)}_{\text{probability of getting at least } \alpha \text{ from } q}}$$

	0	10	20	30
p	$1/8$	$2/8$	$2/8$	$3/8$
q	$1/8$	$3/8$	$3/8$	$1/8$



CDF :



Theorem: p FOSD q $\iff$ $\mathbb{E}_p(u) \geq \mathbb{E}_q(u)$ for all Bernoulli utility functions.

Question C.6: p FOSD $q \implies \mathbb{E}_p(u) \geq \mathbb{E}_q(u)$

	0	10	20	30
p	1/8	2/8	2/8	3/8
q	1/8	3/8	3/8	1/8

$u(0) \leq u(10) \leq u(20) \leq u(30)$

	0	10	20	30
r_1	1/8	2/8	4/8	1/8
r_2	1/8	3/8	3/8	1/8

$\mathbb{E}_p(u) \geq \mathbb{E}_{r_1}(u) \geq \mathbb{E}_{r_2}(u)$
$= \mathbb{E}_q(u)$

$\boxed{p \text{ FOSD } q \Longleftarrow \mathbb{E}_p(u) \geq \mathbb{E}_q(u) \text{ for all Bernoulli utilities}}$

$\boxed{p \text{ \underline{does not} FOSD } q \implies \text{there exists some Bernoulli utility s.t. } \mathbb{E}_p(u) < \mathbb{E}_q(u)}$

I can find some $\alpha \in \mathbb{R}$ such that

$$\sum_{x_k \geq \alpha} p(x_k) < \sum_{y_\ell \geq \alpha} p(y_\ell)$$

Take $u(x) = 0$ for $x < \alpha$
$u(x) = 1$ for $x \geq \alpha$

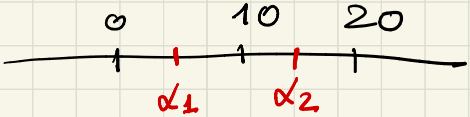
$$\mathbb{E}_p(u) = \sum_{x_k \geq \alpha} p(x_k) < \sum_{y_\ell \geq \alpha} p(y_\ell) = \mathbb{E}_q(u)$$

Not all lotteries can be compared by FOSD.

$$p = \left(\tfrac{1}{2} \times 0, \tfrac{1}{2} \times 20\right)$$

$$q = (1 \times 10)$$

	0	10	20
p	1/2	0	1/2
q	0	1	0



- $u(x) = x$

$$\mathbb{E}_p(u) = 10 = \mathbb{E}_q(u)$$

$$\Longrightarrow \boxed{p \sim q}$$

- $u(x) = x^2$

$$\mathbb{E}_p(u) = \frac{1}{2} 400 = 200$$

$$\mathbb{E}_q(u) = 100$$

$$\Longrightarrow \boxed{p \succ q}$$

vNM AXIOMS:

Completeness and transitivity, plus two additional axioms, continuity and independence.

Theorem: $\succsim$ over $\mathcal{L}$ have a vNM EU representation $\Longleftrightarrow$ $\succsim$ satisfy these four axioms

Question C.7: $\succsim$ have a vNM EU representation. Then, prove:

$x \succ y$ and $\alpha > \beta \Longrightarrow$

$$(\alpha \times x, (1-\alpha) \times y) \succ (\beta \times x, (1-\beta) \times y)$$

$x \succ y \Longrightarrow u(x) > u(y)$

$$\Longrightarrow \alpha u(x) + (1-\alpha) u(y) > \beta u(x) + (1-\beta) u(y)$$

$$\Longrightarrow (\alpha \times x, (1-\alpha) \times y) \succ (\beta \times x, (1-\beta) \times y)$$

Question C.8 : u is a Bernoulli utility function. Then, the following holds:

$$\boxed{v \text{ is also Bernoulli} \iff \boxed{v(x) = \alpha + \beta u(x)},\ \beta > 0}$$

$(\Leftarrow)$ $p \succsim q \iff \sum_{k=1}^{K} p(x_k)\, u(x_k) \geq \sum_{l=1}^{L} q(y_l)\, u(y_l)$

Take $\sum_{k=1}^{K} p(x_k)\, v(x_k) = \sum_{k=1}^{K} p(x_k) \left(\alpha + \beta u(x_k)\right)$

$$= \alpha \underbrace{\sum_{k=1}^{K} p(x_k)}_{1} + \beta \sum_{k=1}^{K} p(x_k)\, u(x_k)$$

$$= \boxed{\alpha + \beta \sum_{k=1}^{K} p(x_k)\, u(x_k)}$$

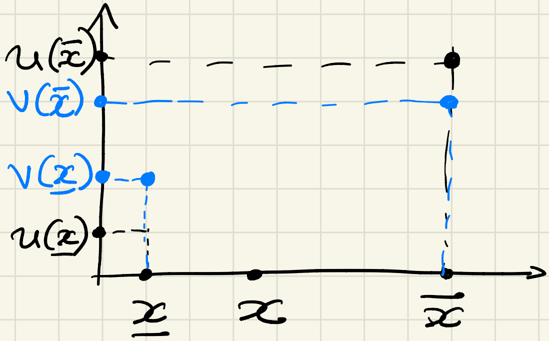
$$\sum_{l=1}^{L} q(y_l)\, v(y_l) = \boxed{\alpha + \beta \sum_{l=1}^{L} q(y_l)\, u(y_l)}$$

$\Rightarrow E_p(v) \geq E_q(v)$.

$(\Longrightarrow)$ Take $\bar{x}$, $\underline{x}$ such that $\bar{x} \succsim x \succsim \underline{x}$ for x. Hence,

$$\left.\begin{aligned} u(\bar{x}) &\geq u(x) \geq u(\underline{x}) \\ v(\bar{x}) &\geq v(x) \geq v(\underline{x}) \end{aligned}\right\} \text{because both } u \text{ and } v \text{ are Bernoulli;}$$

Take $\bar{x} \succ \underline{x}$ (otherwise everything is trivial)



Then there exist $\alpha \in \mathbb{R}$ and $\beta > 0$ s.t.

$$\boxed{\begin{aligned} v(\bar{x}) &= \alpha + \beta u(\bar{x}) \\ v(\underline{x}) &= \alpha + \beta u(\underline{x}) \end{aligned}} \quad (***)$$

(this is because $u(\bar{x}) > u(\underline{x})$ and $v(\bar{x}) > v(\underline{x})$)

Take x, and notice that

$$u(\bar{x}) > u(x) > u(\underline{x}) \Longrightarrow$$

$\Longrightarrow$ there is some $\lambda \in (0,1)$ such that

$$\boxed{u(x) = \lambda u(\bar{x}) + (1-\lambda) u(\underline{x})} \quad (*)$$

$$\Longrightarrow x \sim (\lambda \times \bar{x}, (1-\lambda) \times \underline{x})$$

$$\Longrightarrow \boxed{v(x) = \lambda v(\bar{x}) + (1-\lambda) v(\underline{x})} \quad (*)(*)$$

$$v(x) = \lambda v(\bar{x}) + (1-\lambda) v(\underline{x})$$

$$\overset{***}{=} \lambda\left(\alpha + \beta u(\bar{x})\right) + (1-\lambda)\left(\alpha + \beta u(\underline{x})\right)$$

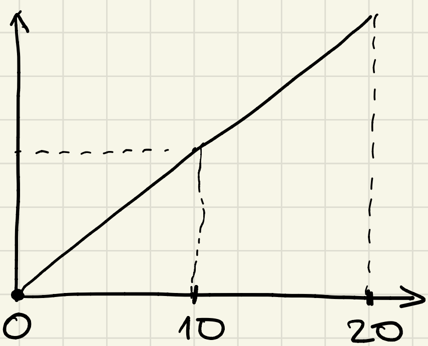
$$= \alpha + \beta\left(\lambda u(\bar{x}) + (1-\lambda) u(\underline{x})\right)$$

$$\overset{*}{=} \alpha + \beta u(x) \quad \checkmark$$

Question C.9: "the utility function is EU" is a cardinal property

$$X = \{0, 10, 20\}$$

$$20 > 10 > 0 \iff \underset{\overset{\parallel}{20}}{u(20)} > \underset{\overset{\parallel}{10}}{u(10)} > \underset{\overset{\parallel}{0}}{u(0)}$$



$$V(p) = \left(\mathbb{E}_p(u)\right)^2$$

$$p \succsim q \iff \sum p(x) u(x) \geq \sum q(y) u(y)$$

$$\iff \left(\sum p(x) u(x)\right)^2 \geq \left(\sum q(y) u(y)\right)^2$$

$$V(p) = \left(p(20) \cdot 20 + p(10) \cdot 10\right)^2$$

$$= 400 \left(p(20)\right)^2 + 100 \left(p(10)\right)^2$$

$$+ 400\, p(10) \cdot p(20)$$

This is not an expected utility.

RISK ATTITUDES:

A lottery $p = (P(x_1)\times x_1, \dots, P(x_K)\times x_K)$ and a sure outcome $x = \sum_{k=1}^{K} P(x_k)\cdot x_k$

Example: $p = (\frac{1}{2}\times 0, \frac{1}{2}\times 20)$, $x = 10$

- RISK AVERSE: $x \succsim p$
- RISK SEEKING: $x \precsim p$
- RISK NEUTRAL: $x \sim p$

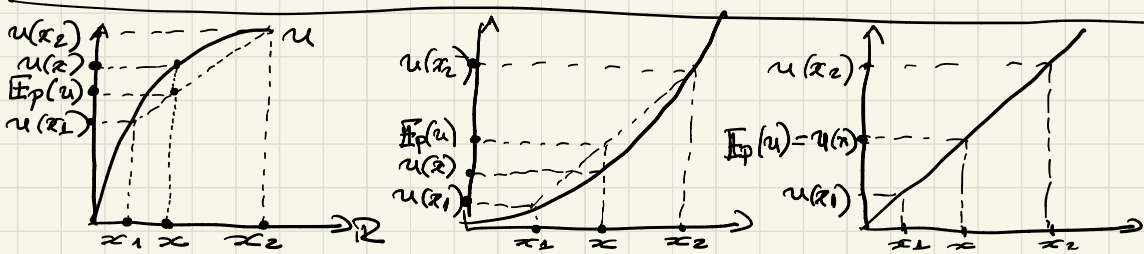
for all lotteries and their correspondent expected-payoff-equivalent outcome.

Theorem: If $\succsim$ have an EU representation (with Bernoulli u):

(i) DM risk-averse (resp. strictly risk averse) $\iff$ u concave (resp. strictly concave)

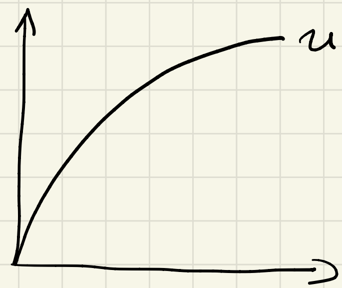
(ii) DM risk-seeking (resp. strictly risk seeking) $\iff$ u convex (resp. strictly convex)

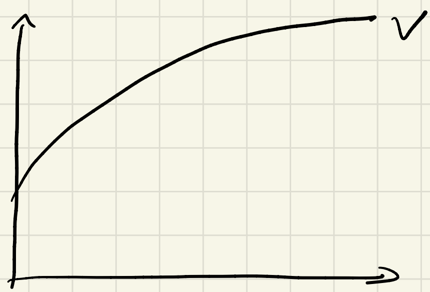
(iii) DM risk-neutral $\iff$ u linear



If $\succsim$ have an EU representation the previous theorem holds no matter which Bernoulli utility function I use.

Recall from Q. C.8 that Bernoulli utility functions are always strictly positive transformations of one another. And such transformations preserve convexity.

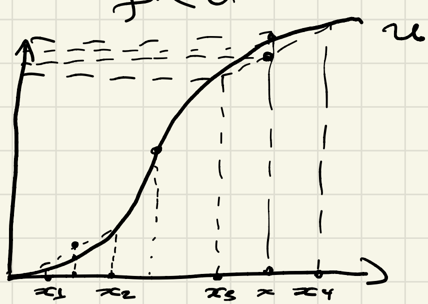




$$\boxed{v = \alpha + \beta u \quad , \beta > 0}$$

$$u \text{ concave} \iff \alpha + \beta u \text{ concave} \iff v \text{ concave}$$

Question C.10: DM is not necessarily classified into one of the three types



Question C.11 :

$$p \succsim q \iff \underbrace{\sum_{k=1}^{K} p(x_k)\cdot x_k}_{\mathbb{E}_p(u)} \geq \underbrace{\sum_{l=1}^{L} q(y_l)\, y_l}_{\mathbb{E}_q(u)}$$

(given $u(x)=x$)

But then $u(x)=x$ is linear.

By Theorem C.3, DM is risk-neutral.

Question C.12 : "The DM is risk-averse" is ordinal.

$\boxed{p \precsim x}$ (where $x = \sum_{k=1}^{K} p(x_k) x_k$).

Take a EU utility function of a risk averse DM : $u(x) \geq \mathbb{E}_p(u) = \sum_{k=1}^{K} p(x_k)\, u(x_k)$

Take strictly increasing $f: \mathbb{R} \to \mathbb{R}$ and

$$\left.\begin{aligned} V(p) &= f(\mathbb{E}_p(u)) \\ V(x) &= f(u(x)) \end{aligned}\right\} \quad V(x) \succsim V(p)$$

$\Longrightarrow x \succsim p$.

SECOND ORDER STOCHASTIC DOMINANCE

$$\boxed{P \text{ SOSD } q \iff \mathbb{E}_p(u) \geq \mathbb{E}_q(u) \text{ for every concave } u}$$

Question C.13: $P = \left(\frac{1}{2} \times 30, \frac{1}{2} \times 10\right)$

$$q = \left(\frac{1}{4} \times 40, \frac{1}{2} \times 20, \frac{1}{4} \times 0\right)$$

u concave whenever for all x, y and all $\lambda \in (0,1)$:

$$(*) \quad \boxed{u(\lambda x + (1-\lambda) y) \geq \lambda u(x) + (1-\lambda) u(y)}$$

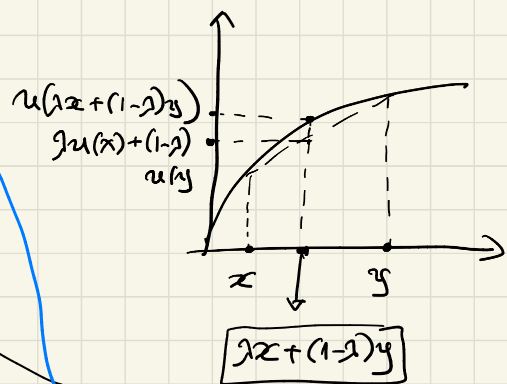
$$\boxed{\mathbb{E}_p(u) = \frac{1}{2} \boxed{u(30)} + \frac{1}{2} u(10)}$$

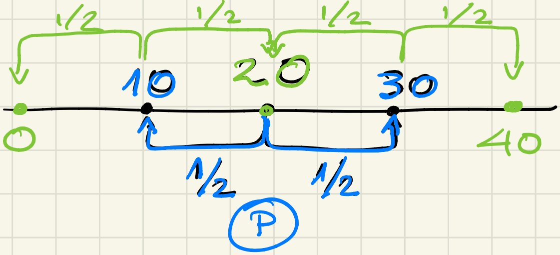
$$\mathbb{E}_q(u) = \frac{1}{4} u(40) + \boxed{\frac{1}{2} u(20)} + \frac{1}{4} u(0)$$

$$= \frac{1}{4} u(40) + \boxed{\frac{1}{4} u(20)} + \boxed{\frac{1}{4} u(20)} + \frac{1}{4} u(0)$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{2} u(40) + \frac{1}{2} u(20)}\right) + \frac{1}{2}\left(\underbrace{\frac{1}{2} u(20) + \frac{1}{2} u(0)}\right)$$

$$\leq \frac{1}{2} u\left(\frac{1}{2} 40 + \frac{1}{2} 20\right) + \frac{1}{2} u\left(\frac{1}{2} 20 + \frac{1}{2} 0\right) \leq \boxed{\frac{1}{2} u(30) + \frac{1}{2} u(10)}$$





Question C.14:

p FOSD $q \iff E_p(u) \geq E_q(u)$ for all u

$\implies E_p(u) \geq E_q(u)$ for all concave u

$\iff p$ SOSD q

$p = (1 \times 10)$ $\qquad$ $q = \left(\frac{1}{2} \times 20, \frac{1}{2} \times 0\right)$

p SOSD q : Take a concave u

$$E_p(u) = u(10) = u\left(\frac{1}{2} \cdot 20 + \frac{1}{2} \cdot 0\right)$$

$$\geq \frac{1}{2} u(20) + \frac{1}{2} u(0)$$

$$= E_q(u)$$

Take some u strictly convex

$$E_p(u) < E_q(u)$$

meaning that p does not FOSD q.

RATIONALIZING CHOICES UNDER UNCERTAINTY

$\mathcal{E}$ contains $A \subseteq \mathcal{L}$

$C(A) \subseteq A$ choice correspondence

It is rationalized if there is a Bernoulli $u : X \to \mathbb{R}$ such that

$$\boxed{C(A) = \{p \in A : \mathbb{E}_p(u) \geq \mathbb{E}_q(u) \text{ for all } q \in A\}}$$

for every $A \in \mathcal{E}$.

Question C15:

$p_1 = (0.25 \times 3000,\ 0.75 \times 0)$ $p_2 = (0.2 \times 4000,\ 0.8 \times 0)$	$C(\{p_1, p_2\}) = \{p_2\}$
$q_1 = (1 \times 3000)$ $q_2 = (0.8 \times 4000,\ 0.2 \times 0)$	$C(\{q_1, q_2\}) = \{q_1\}$

Suppose there is a Bernoulli utility function:

$$\mathbb{E}_{p_2}(u) = \frac{1}{5} u(4000) + \frac{4}{5} u(0) > \frac{1}{4} u(3000) + \frac{3}{4} u(0) = \mathbb{E}_{p_1}(u)$$

$$\mathbb{E}_{q_1}(u) = u(3000) > \frac{4}{5} u(4000) + \frac{1}{5} u(0) = \mathbb{E}_{q_2}(u)$$

$$\frac{1}{5}u(4000)+\frac{4}{5}u(0) > \frac{1}{4}u(3000)+\frac{3}{4}u(0)$$

$$\Rightarrow \boxed{\frac{4}{5}u(4000)+\frac{16}{5}u(0) > u(3000)+3u(0)}$$

$$u(3000) > \frac{4}{5}u(4000)+\frac{1}{5}u(0)$$

$$\Rightarrow \boxed{u(3000)+3u(0) > \frac{4}{5}u(4000)+\frac{16}{5}u(0)}$$

contradiction, meaning that vNM axioms are rejected.

Question C.16 :

$P_1 = (1 \times 10)$
$P_2 = (0.5 \times 20, 0.5 \times 0)$

$C(\{P_1, P_2\}) = \{P_1, P_2\}$ ✓

$q_1 = (1 \times 20)$
$q_2 = (\frac{1}{3} \times 80, \frac{2}{3} \times 10)$

$C(\{q_1, q_2\}) = \{q_1, q_2\}$ ✓

$r_1 = (1 \times 80)$
$r_2 = (\frac{1}{3} \times 160, \frac{2}{3} \times 20)$

$C(\{r_1, r_2\}) = \{r_1, r_2\}$ ✓

$$P_1 \sim P_2 \iff u(10) = \frac{1}{2} u(20) + \frac{1}{2} u(0)$$

Take $\boxed{u(0) = 0} \Rightarrow \boxed{u(20) = 2u(10)}$

$$q_1 \sim q_2 \iff u(20) = \frac{1}{3} u(80) + \frac{2}{3} u(10)$$
$$\iff 2u(10) = \frac{1}{3} u(80) + \frac{2}{3} u(10)$$
$$\iff \boxed{u(80) = 4u(10)}$$

$$r_1 \sim r_2 \iff u(80) = \frac{1}{3} u(160) + \frac{2}{3} u(20)$$
$$\iff 4u(10) = \frac{1}{3} u(160) + \frac{4}{3} u(10)$$
$$\iff 12u(10) = u(160) + 4u(10)$$
$$\iff \boxed{u(160) = 8u(10)}$$

Take $\boxed{u(10) = 10} \Rightarrow \boxed{u(20) = 20}, \boxed{u(80) = 40}, \boxed{u(160) = 80}$

SUBJECTIVE UNCERTAINTY :

States : $\Omega = \{\omega_1, \dots, \omega_N\}$

Examples : • $\Omega = \{\underbrace{D}_{\text{democrat}}, \underbrace{R}_{\text{republican}}\}$

• $\Omega = \{\underbrace{G}_{\text{no accident}}, \underbrace{B}_{\text{accident}}\}$

Acts : $f : \Omega \longrightarrow \mathcal{L}$

$$f = (\underbrace{f(\omega_1)}_{\text{if } \omega_1 \text{ occurs}}, \underbrace{f(\omega_2)}_{\text{if } \omega_2 \text{ occurs}}, \dots, \underbrace{f(\omega_N)}_{\text{if } \omega_N \text{ occurs}}) \quad \text{contract}$$

$$f = (P_1, P_2, \dots, P_N)$$

Examples : • $\Omega = \{D, R\}$

$$f = (f(D), f(R)) \qquad f(D) = (1 \times 9) \qquad f(R) = (1 \times -1)$$

I bet €1 into D in which case I will get €10 back.

• $\Omega = \{G, B\}$

$$g = (g(G), g(B)) \qquad g(G) = (1 \times -100) \qquad g(B) = (0.01 \times -2100, \; 0.99 \times -1100)$$

I buy insurance for €100; if there is an accident, the company will assign an expert who approves coverage of €1000 with prob .99. Assume that the car damage is €2000.

• $\mathcal{F}$ set of all acts

• $(P, \dots, P) = \boxed{P}$ constant act : $\mathcal{L} \subseteq \mathcal{F}$

PREFERENCES over $\mathcal{F}$

$f \succsim g$: "f at least as good as g"

$\succ$ and $\sim$ are defined in the usual way

The induced preferences over $\mathcal{L}$:

$$\boxed{p \succsim q \iff (p, \dots, p) \succsim (q, \dots, q)}$$

The induced preferences over $\mathcal{L}$ have an EU representation (there is a Bernoulli utility function u).

When I have an act $f = (\underset{\boxed{\omega_1}}{P_1}, \dots, \underset{\boxed{\omega_N}}{P_N})$

I get is a vector of EU's : $(E_{P_1}(u), \dots, E_{P_N}(u))$

Question C.17: $\Omega = \{\text{Democrat}(\omega_1), \text{Republican}(\omega_2)\}$

$$\boxed{f} \succsim \boxed{g} \iff \min\{\underbrace{\mathbb{E}_{\boxed{f(\omega_1)}}(u)}_{\substack{\text{expected utility if}\\ f \text{ is chosen}\\ \text{and Democrat wins}}}, \underbrace{\mathbb{E}_{\boxed{f(\omega_2)}}(u)}_{\substack{\text{expected utility if}\\ f \text{ is chosen}\\ \text{and Republican wins}}}\} \geq \min\{\underbrace{\mathbb{E}_{\boxed{g(\omega_1)}}(u)}_{\substack{\text{EU if}\\ g \text{ chosen}\\ \text{and democrat wins}}}, \underbrace{\mathbb{E}_{\boxed{g(\omega_2)}}(u)}_{\substack{\text{EU if}\\ g \text{ chosen}\\ \text{and republican wins}}}\}$$

See Questions C.3 and C.4.

Question C.18: $\Omega = \{\omega_1, \omega_2\}$

$$f \succsim g \iff f(\omega_1) \succsim g(\omega_1)$$

Example:
$$\left.\begin{array}{l} f = (\boxed{1}, 0) \\ g = (\boxed{0}, 1000) \end{array}\right\} f(\omega_1) \succ g(\omega_1) \implies f \succ g$$

See Questions C.3 and C.4.

$$p \succsim q \iff (\boxed{p}, p) \succsim (\boxed{q}, q)$$
$$\iff \mathbb{E}_p(u) \geq \mathbb{E}_q(u)$$

UTILITY REPRESENTATION : $u : \mathcal{F} \to \mathbb{R}$

$$\boxed{f \succsim g \iff u(f) \geq u(g)}$$

SUBJECTIVE BELIEFS: $\Omega = \{\omega_1, \dots, \omega_N\}$

A probability distribution μ over Ω

So $\mu(\omega)$: subjective probability of ω occuring.

- $\mu(\omega_1) + \dots + \mu(\omega_N) = 1$
- μ is unobservable by economist.

SUBJECTIVE EXPECTED UTILITY:

- First take Bernoulli utility and construct vector of expected utilities:

$$\boxed{f} = (\underset{\boxed{\omega_1}}{P_1}, \dots, \underset{\boxed{\omega_N}}{P_N}) \Longrightarrow \boxed{u(f)} = (\underset{\boxed{\omega_1}}{\mathbb{E}_{P_1}(u)}, \dots, \underset{\boxed{\omega_N}}{\mathbb{E}_{P_N}(u)})$$

(For a state ω, $\mathbb{E}_{f(\omega)}(u)$ this is the corresponding EU.)

- Second, take the subjective belief and compute SEU:

$$u(f) = \left(\underset{\substack{\boxed{\omega_1} \\ \boxed{\mu(\omega_1)}}}{\boxed{\mathbb{E}_{\boxed{P_1}}(u)}}, \dots, \underset{\substack{\boxed{\omega_N} \\ \boxed{\mu(\omega_N)}}}{\boxed{\mathbb{E}_{\boxed{P_N}}(u)}}\right) \Longrightarrow \mathbb{E}_\mu(u(f)) =$$

$$= \mu(\omega_1) \cdot \mathbb{E}_{\boxed{f(\omega_1)}}(u) + \dots + \mu(\omega_N) \cdot \mathbb{E}_{\boxed{f(\omega_N)}}(u)$$

$$\underbrace{\mathbb{E}_\mu\left(u(f(\omega))\right)}_{\substack{\text{subjective EU from} \\ \text{act } f\text{, given Bernoulli} \\ \text{utility } u \text{ and beliefs } \mu}} = \sum_{\omega\in\Omega} \underbrace{\mu(\omega)}_{\substack{\text{belief} \\ \text{of } \omega \\ \text{occuring}}} \underbrace{\mathbb{E}_{f(\omega)}(u)}_{\substack{\text{objective expected} \\ \text{utility from the} \\ \text{lottery that } f \text{ gives} \\ \text{me at each state} \\ \omega\in\Omega.}}$$

weighted average of objective expected utilities

SEU representation (AA): $\succsim$ over $\mathcal{F}$ have an AA SEU representation if we can find Bernoulli utility u and beliefs μ, such that:

$$\boxed{f \succsim g \iff \mathbb{E}_\mu(u(f)) \geq \mathbb{E}_\mu(u(g))}$$

Question C.19:

$$f \succsim g \iff \min\{\mathbb{E}_{f(\omega_1)}(u), \mathbb{E}_{f(\omega_2)}(u)\} \geq \min\{\mathbb{E}_{g(\omega_1)}(u), \mathbb{E}_{g(\omega_2)}(u)\}$$

$f = (1, 0) \qquad g = (0, 1)$

Let $u(1) > u(0)$

$$\left.\begin{array}{l} u(f) = (u(1), u(0)) \\ u(g) = (u(0), u(1)) \end{array}\right\} \Rightarrow \boxed{f \sim g} \text{ because}$$

$$\min\{u(1), u(0)\} = u(0) = \min\{u(0), u(1)\}.$$

If there is SEU representation, then

$$\mu(\omega_1)u(1) + \mu(\omega_2)u(0) = \mu(\omega_1)u(0) + \mu(\omega_2)u(1)$$

$$\Rightarrow \boxed{\mu(\omega_1) = \mu(\omega_2) = 1/2}$$

$$h = (100, 0) \Rightarrow u(h) = (u(100), u(0))$$

$$\mathbb{E}_\mu(u(h)) = \tfrac{1}{2}u(100) + \tfrac{1}{2}u(0) > \mathbb{E}_\mu(u(f)) \quad (*)$$

$$\min\{u(100), u(0)\} = u(0) \quad (**)$$

$$\left.\begin{array}{l} (*) \Rightarrow h \succ f \\ (**) \Rightarrow h \sim f \end{array}\right\} \text{contradiction.}$$

$$f \succsim g \iff f(\omega_1) \succsim g(\omega_1)$$

$$\mu(\omega_1) = 1 \implies \left.\begin{aligned} \mathbb{E}_\mu\big(u(f)\big) &= \mathbb{E}_{f(\omega_1)}(u) \\ \mathbb{E}_\mu\big(u(g)\big) &= \mathbb{E}_{g(\omega_1)}(u) \end{aligned}\right\} \implies$$

$$\implies f \succsim g \iff \mathbb{E}_{f(\omega_1)}(u) \geq \mathbb{E}_{g(\omega_1)}(u)$$

$$\iff f(\omega_1) \succsim g(\omega_1) \quad \boxed{\text{done}}.$$

AA AXIOMS: Completeness, Transitivity, Continuity, Independence, plus the following two axioms: AA monotonicity and non-triviality.

$$\boxed{\text{Theorem: } \succsim \text{ over } \mathcal{F} \text{ have a SEU representation} \iff \text{they satisfy the previous six axioms}}$$

DOMINANCE RELATIONS:

$$f = \left(\boxed{p_1}, \dots, \boxed{p_N}\right) \Longrightarrow u(f) = \left(\boxed{\mathbb{E}_{p_1}(u)}, \dots, \boxed{\mathbb{E}_{p_N}(u)}\right)$$

$$g = \left(\boxed{q_1}, \dots, \boxed{q_N}\right) \Longrightarrow u(g) = \left(\boxed{\mathbb{E}_{q_1}(u)}, \dots, \boxed{\mathbb{E}_{q_N}(u)}\right)$$

STRICT DOMINANCE:

f strictly dominates $g \iff f(\omega) \succ g(\omega)$ for all $\omega \in \Omega$

$\iff \mathbb{E}_{f(\omega)}(u) > \mathbb{E}_{g(\omega)}(u)$ for all $\omega \in \Omega$

WEAK DOMINANCE:

f weakly dominates $g \iff f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, with at least one preference being strict

$\iff \mathbb{E}_{f(\omega)}(u) \geq \mathbb{E}_{g(\omega)}(u)$ for all $\omega \in \Omega$, with at least one inequality being strict

Question C.20: f strictly dominates $g \Rightarrow f \succ g$ (irrespective of beliefs).

$$f(\omega) \succ g(\omega) \iff \mathbb{E}_{f(\omega)}(u) > \mathbb{E}_{g(\omega)}(u) \quad \text{for all } \omega \in \Omega$$

$$\Rightarrow \boxed{\mu(\omega)\,\mathbb{E}_{f(\omega)}(u) \geq \mu(\omega)\,\mathbb{E}_{g(\omega)}(u)} \quad \text{for all } \omega \in \Omega$$

with strict inequality iff $\mu(\omega) > 0$

$$\Rightarrow \sum_{\omega \in \Omega} \mu(\omega)\,\mathbb{E}_{f(\omega)}(u) > \sum_{\omega \in \Omega} \mu(\omega)\,\mathbb{E}_{g(\omega)}(u)$$

$$\iff \mathbb{E}_\mu(u(f)) > \mathbb{E}_\mu(u(g))$$

$$\iff f \succ g$$

Take $u(0) = 0$, $u(1) = 7$, $u(2) = 10$

$$f = (1, 1) \Rightarrow u(f) = (7, 7)$$

$$g = (2, 0) \Rightarrow u(g) = (10, 0)$$

Take $\mu(\omega_1) = \mu(\omega_2) = 1/2$

$$\mathbb{E}_\mu(u(f)) = 7 > 5 = \frac{1}{2}10 + \frac{1}{2}0 = \mathbb{E}_\mu(u(g))$$

Still f does not strictly dominate g.

Question C.21: $\left.\begin{array}{l}\cdot\ f \text{ weakly dominates } g \\ \cdot \text{ beliefs are full support}\end{array}\right\} \Rightarrow f \succ g$

$f(\omega) \succsim g(\omega)$ with at least one strict preference $\iff \mathbb{E}_{f(\omega)}(u) \geq \mathbb{E}_{g(\omega)}(u)$ with at least one inequality being strict

$\xRightarrow{\text{(full support)}} \mu(\omega)\mathbb{E}_{f(\omega)}(u) \geq \mu(\omega)\mathbb{E}_{g(\omega)}(u)$ with at least one inequality strict

$\Rightarrow \sum_{\omega\in\Omega} \mu(\omega)\mathbb{E}_{f(\omega)}(u) > \sum_{\omega\in\Omega}\mu(\omega)\mathbb{E}_{g(\omega)}(u)$

$\iff \mathbb{E}_\mu(u(f)) > \mathbb{E}_\mu(u(g))$

$\iff f \succ g$

$\boxed{u(0)=0,\ u(1)=10}$

$f=(1,1) \Rightarrow u(f) = (10, 10)$

$g=(1,0) \Rightarrow u(g) = (10, 0)$

f weakly dominates g.

Take $\boxed{\mu(\omega_1)=1}$. Then $\mathbb{E}_\mu(u(f)) = 10 = \mathbb{E}_\mu(u(g))$

$\Rightarrow f \sim g$.

MIXED ACTS: $f, g \in \mathcal{F}$, then I can take the mixed act $(\alpha \times f, (1-\alpha) \times g)$, that pays $\alpha f(\omega) + (1-\alpha) g(\omega)$ at each state ω.

Example:

	ω_1	ω_2
f	(1×10)	$(\frac{1}{2} \times 20, \frac{1}{2} \times 0)$
g	(1×0)	(1×20)
$(\frac{1}{2} \times f, \frac{1}{2} \times g)$	$(\frac{1}{2} \times 10, \frac{1}{2} \times 0)$	$(\frac{3}{4} \times 20, \frac{1}{4} \times 0)$

Question C.22 :

	ω_1	ω_2
f	10	0
g	0	10
h	5	5

$\Longrightarrow$

		ω_1	ω_2
$\boxed{\alpha}$	f	$u(10)$	$u(0)$
$\boxed{1-\alpha}$	g	$u(0)$	$u(10)$
	h	$u(5)$	$u(5)$
		$\mu(\omega_1)$	$\mu(\omega_2)$

unobservables.

(a) Risk seeking $\Rightarrow$ u strictly convex

$$\Rightarrow u\left(\tfrac{1}{2}10+\tfrac{1}{2}\cdot 0\right) < \tfrac{1}{2}u(10)+\tfrac{1}{2}u(0)$$

$$\underset{u(5)}{\shortparallel}$$

	ω_1	ω_2
$\alpha f + (1-\alpha) g$	$\alpha u(10)+(1-\alpha)u(0)$	$\alpha u(0)+(1-\alpha)u(10)$
h	$u(5)$	$u(5)$

If I take $\alpha = 1/2$, then

$$\boxed{\tfrac{1}{2}u(10)+\tfrac{1}{2}u(0) > u(5)}$$

$$\Rightarrow \mathbb{E}_{(\alpha f+(1-\alpha)g)(\omega)}(u) > \mathbb{E}_{h(\omega)}(u)$$

$\Rightarrow$ $\alpha f+(1-\alpha)g$ strictly dominates h.

(b)

	ω_1	ω_2
f	10	0
g	0	10
h	5	5

$\Longrightarrow$

	ω_1	ω_2
f	$u(10)$	$u(0)$
g	$u(0)$	$u(10)$
h	$u(5)$	$u(5)$
	$\mu(\omega_1)$	$\mu(\omega_2)$

unobservables

Risk. aversion $\Longrightarrow$ $\boxed{u(5) > \frac{1}{2}u(10) + \frac{1}{2}u(0)}$

For some μ:

$$\begin{cases} \mathbb{E}_\mu(u(h)) \geq \mathbb{E}_\mu(u(f)) \\ \mathbb{E}_\mu(u(h)) \geq \mathbb{E}_\mu(u(g)) \end{cases}$$

Take $\mu(\omega_1) = \mu(\omega_2) = 1/2$

$$\boxed{\begin{aligned} \mathbb{E}_\mu(u(h)) &= \frac{1}{2}u(5) + \frac{1}{2}u(5) = u(5) \\ \mathbb{E}_\mu(u(f)) &= \frac{1}{2}u(10) + \frac{1}{2}u(0) \\ \mathbb{E}_\mu(u(g)) &= \frac{1}{2}u(0) + \frac{1}{2}u(10) \end{aligned}}$$

RATIONALIZING CHOICE UNDER SUBJECTIVE UNCERTAINTY

C is rationalized by an AA SEU function if there exist Bernoulli utilities u and beliefs μ, such that:

$$\boxed{C(A) = \{f \in A : \mathbb{E}_\mu(u(f)) \geq \mathbb{E}_\mu(u(g)) \text{ for all } g \in A\}}$$

for every $A \in \mathcal{E}$.

Question C.23: An urn contains 3 balls, exactly one red, the remaining balls are either (i) both black, or (ii) one black and one yellow, or (iii) both yellow. Then we randomly draw a ball and depending on which act the subject has chosen, we will pay accordingly:

f_1: if red then €10 f_2: if black then €10	$C(\{f_1, f_2\}) = \{f_1\}$
g_1: if red or yellow then €10 g_2: if black or yellow then €10	$C(\{g_1, g_2\}) = \{g_2\}$

ω_1 : both black

ω_2 : one black one yellow

ω_3 : both yellow

	ω_1	ω_2	ω_3
f_1	$(\frac{1}{3}\times 10, \frac{2}{3}\times 0)$	$(\frac{1}{3}\times 10, \frac{2}{3}\times 0)$	$(\frac{1}{3}\times 10, \frac{2}{3}\times 0)$
f_2	$(\frac{2}{3}\times 10, \frac{1}{3}\times 0)$	$(\frac{1}{3}\times 10, \frac{2}{3}\times 0)$	(1×0)
g_1	$(\frac{1}{3}\times 10, \frac{2}{3}\times 0)$	$(\frac{2}{3}\times 10, \frac{1}{3}\times 0)$	(1×10)
g_2	$(\frac{2}{3}\times 10, \frac{1}{3}\times 0)$	$(\frac{2}{3}\times 10, \frac{1}{3}\times 0)$	$(\frac{2}{3}\times 10, \frac{1}{3}\times 0)$

Suppose there is an AA SEU representation:

$3 = u(10) > u(0) = 0$

	ω_1	ω_2	ω_3
f_1	1	1	1
f_2	2	1	0
g_1	1	2	3
g_2	2	2	2
	$\mu(\omega_1)$	$\mu(\omega_2)$	$\mu(\omega_3)$

$$\mathbb{E}_\mu(u(f_1)) = \mu(\omega_1) + \mu(\omega_2) + \mu(\omega_3) = 1$$

$$\mathbb{E}_\mu(u(f_2)) = 2\mu(\omega_1) + \mu(\omega_2)$$

$$\mathbb{E}_\mu(u(g_1)) = \mu(\omega_1) + 2\mu(\omega_2) + 3\mu(\omega_3)$$

$$\mathbb{E}_\mu(u(g_2)) = 2\mu(\omega_1) + 2\mu(\omega_2) + 2\mu(\omega_3) = 2$$

$$1 > 2\mu(\omega_1) + \mu(\omega_2) \Longrightarrow \boxed{\mu(\omega_2) < 1 - 2\mu(\omega_1)}$$

$$2 > \mu(\omega_1) + 2\mu(\omega_2) + 3\big(1 - \mu(\omega_1) - \mu(\omega_2)\big)$$

$$= 3 - 2\mu(\omega_1) - \mu(\omega_2)$$

$$\Longrightarrow \boxed{\mu(\omega_2) > 1 - 2\mu(\omega_1)}$$

contradiction.

GAME THEORY: Multiple decision makers interacting (one's action affects the others)

STRATEGIC FORM GAMES: $\langle I, (A_i)_{i\in I}, (o_i)_{i\in I} \rangle$

- $I = \{\text{Ann } (a), \text{Bob } (b)\}$ set of players (i, j)
- For each $i \in I$, A_i is the set of actions of player i.
- For each profile of actions (a_i, a_j) or (a_a, a_b)

$$o_i(a_i, a_j) \in X \subseteq \mathbb{R}$$

$$\Big(\underbrace{O_a(\underbrace{a_a, a_b}_{\text{profile of actions}})}_{\text{outcome for player } a.}, \underbrace{O_b(\underbrace{a_a, a_b}_{\text{profile of actions}})}_{\text{outcome for player } b} \Big) \in \mathbb{R}^2$$

Example D.1: $A_a = \{T, M, B\}$ $A_b = \{L, R\}$

	L	R
T	10, 5	0, 0
M	0, 5	10, 0
B	5, 0	5, 5

$$\boxed{\begin{array}{l} O_a(M, L) = 0 \\ O_b(M, L) = 5 \end{array}}$$

GAMES can be seen as choice problems under subjective uncertainty (for each player).

	L	R
T	10,5	0,0
M	0,5	10,0
B	5,0	5,5

Let for a minute put ourselves into Ann's shoes.

From her points of view

Bob's actions can be seen as states

over which she has subjective uncertainty.

Her own actions can be seen as acts

PREFERENCES : Assume that each player has AA SEU preferences over acts.

- Bernoulli utility functions are defined over pairs of monetary payoffs

$$(a_i, a_j) \succsim_i (a_i', a_j') \Leftrightarrow \left(o_i(a_i,a_j), o_j(a_i,a_j)\right) \succsim_i \left(o_i(a_i',a_j'), o_j(a_i',a_j')\right)$$

Question D.1 :

	L	R
T	10,5	0,0
M	0,5	10,0
B	5,0	5,5

RISK-NEUTRAL

$\boxed{u_i(x) = x}$

	L	R
T	10,5	0,0
M	0,5	10,0
B	5,0	5,5

RISK-AVERSE

$\boxed{u_i(0)=0,\ u_i(10)=10,\ u_i(5)=6}$

	L	R
T	10,6	0,0
M	0,6	10,0
B	6,0	6,6

RISK-SEEKING

$\boxed{u_i(0)=0,\ u_i(10)=10,\ u_i(5)=4}$

	L	R
T	10,4	0,0
M	0,4	10,0
B	4,0	4,4

All three utility function induce the same preferences over action profiles.

$$(T, L) \succ_a (B, R)$$

irrespective of which utility function we use (among the three).

SUBJECTIVE EXPECTED UTILITY:

Given i's Bernoulli utility u_i, and beliefs μ_i over the opponent's actions each action $a_i \in A_i$ has SEU:

$$\boxed{\mathbb{E}_{\mu_i}(u_i(a_i)) = \sum_{a_j \in A_j} \mu_i(a_j) \cdot u_i(a_i, a_j)}$$

$$\boxed{a_i \succsim_i a_i' \iff \mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i'))}$$

Clearly, which action is preferred over which depends on Bernoulli utilities and beliefs of the player

RATIONALITY: given μ_i and u_i, an action $a_i \in A_i$ is rational if:

$$\boxed{\mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i'))} \quad (*)$$

for any $a_i' \in A_i$.

Assuming that the economist only knows u_i, then a_i can be rationally chosen if there exists some μ_i such that $(*)$ holds

Question D.2: I only write the utilities of Ann, and for all three cases I have $u_a(10)=10$, $u_a(0)=0$, and

	L	R
T	10	0
M	0	10
B	x	x

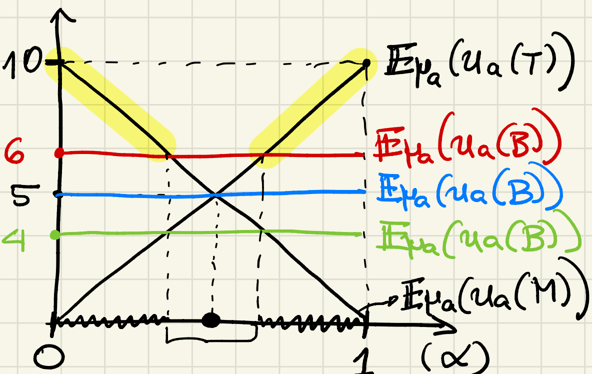
$$\underbrace{\mu_a(L)}_{\alpha} + \underbrace{\mu_a(R)}_{1-\alpha} = 1$$

- Risk neutral : $u_a(5)=5$
- Risk averse : $u_a(5)=6$
- Risk seeking : $u_a(5)=4$

$\left(x = u_a(5)\right)$

$$\left.\begin{aligned} E_{\mu_a}(u_a(T)) &= 10\cdot\alpha \\ E_{\mu_a}(u_a(M)) &= 10(1-\alpha) = 10-10\alpha \\ E_{\mu_a}(u_a(B)) &= x\alpha + x(1-\alpha) = x \end{aligned}\right\}$$

depend on beliefs (viz. on α) and Bernoulli utilities (viz., on x).



$x=5 \Rightarrow$ T, M, B rational
$x=6 \Rightarrow$ T, M, B rational
$x=4 \Rightarrow$ T, M rational

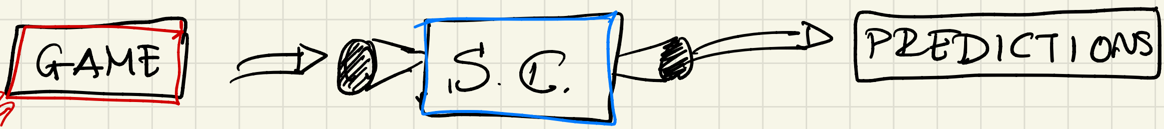
- $\alpha=0 \Rightarrow$ M is rational
- $\alpha=1 \Rightarrow$ T is rational

$$\left.\begin{aligned} &\bullet\ x \geq 10\alpha \\ &\ \ x \geq 10-10\alpha \end{aligned}\right\}\ 2x \geq 10 \iff \boxed{x \geq 5} \Rightarrow$$ B is rational if $x=5$ or $x=6$.

METHODOLOGY:

- ORDINAL GAMES: others only know the induced preferences over action profiles
- CARDINAL GAMES: others know the Bernoulli utility function.

SOLUTION CONCEPTS:



Predictions are action profiles

	ELIMINATION	EQUILIBRIUM
ORDINAL	Iterated Strict/Weak/Börgers Dominance	Nash Equilibrium
CARDINAL	Iterated strict dominance	Mixed NE

ORDINAL GAMES:

$\boxed{u_i : A_i \times A_j \to \mathbb{R}}$ represents preferences over outcomes of the game (over action profiles).

STRICT DOMINANCE: a_i strictly dominates a_i' in the game $A_i \times A_j$, if for all $a_j \in A_j$

$$\boxed{u_i(a_i, a_j) > u_i(a_i', a_j).}$$

	L	M	R
T	3	2	1
M	1	2	3
B	2	1	0

T strictly dominates B

(we do pairwise comparisons between two actions of Ann for every action of Bob)

If there is a_i strictly dominating a_i', then we say that a_i' is strictly dominated

If a_i strictly dominates every other a_i', we say that a_i is strictly dominant

For strict dominance, not knowing Bernoulli utilities is not such a big issue.

Strict dominance is defined given $A_i \times A_j$. Whenever it is obvious which game I am referring to, I will not need to be explicit

Question D.3 : $\boxed{a_i \text{ strictly dominated} \Rightarrow a_i \text{ not rational}}$

Player i maximizes SEU (given the unobservable u_i and μ_i). So, we say that a_i is not rational if there is no Bernoulli utility function and no belief such that

(*) $\mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i'))$ for all $a_i' \in A_i$.

Since a_i is strictly dominated by a_i' :

$$u_i(a_i', a_j) > u_i(a_i, a_j) \quad \text{for all } a_j \in A_j$$

Take any belief μ_i, and then : Question C.20

$$\mu_i(a_j)\, u_i(a_i', a_j) \geq \mu_i(a_j)\, u_i(a_i, a_j)$$

with at least one inequality being strict.

Hence, add across A_j to obtain

$$\underbrace{\sum_{a_j \in A_j} \mu_i(a_j)\, u_i(a_i', a_j)}_{\mathbb{E}_{\mu_i}(u_i(a_i'))} > \underbrace{\sum_{a_j \in A_j} \mu_i(a_j)\, u_i(a_i, a_j)}_{\mathbb{E}_{\mu_i}(u_i(a_i))}$$

Question D.4 : $\boxed{a_i \text{ strictly dominant} \Rightarrow a_i \text{ the only rational action}}$

a_i strictly dominant $\Rightarrow$

a_i strictly dominates every other $a_i' \Rightarrow$

$u_i(a_i, a_j) > u_i(a_i', a_j)$ for all $a_j \in A_j \Longrightarrow$

$\mu_i(a_j)\, u_i(a_i, a_j) \geq \mu_i(a_j)\, u_i(a_i', a_j) \Longrightarrow$
with at least one strict inequality

$$\sum_{a_j \in A_j} \mu_i(a_j)\, u_i(a_i, a_j) > \sum_{a_j \in A_j} \mu_i(a_j)\, u_i(a_i', a_j) \Rightarrow$$

$$\boxed{\mathbb{E}_{\mu_i}(u_i(a_i)) > \mathbb{E}_{\mu_i}(u_i(a_i'))}$$

for all a_i' other than a_i, and all μ_i.

Question D.5: $A_i = \{0, 1, \dots, 10\}$

(how much player i puts in the private account).

a_i : private account

$10 - a_i$: public account

Monetary payoff of i: $a_i + \frac{3}{4}(10 - a_i + 10 - a_j)$

$$= 15 + \frac{1}{4} a_i - \frac{3}{4} a_j$$

$$u_i(a_i, a_j) = 15 + \frac{1}{4} a_i - \frac{3}{4} a_j$$

$a_i = 10$ strictly dominant.

Take $a_i' < 10$

$$u_i(10, a_j) - u_i(a_i', a_j) = \cancel{15} + \frac{1}{4} 10 - \cancel{\frac{3}{4} a_j} - \cancel{15} - \frac{1}{4} a_i' + \cancel{\frac{3}{4} a_j}$$

$$= \frac{1}{4}(10 - a_i') > 0$$

$$\Longrightarrow \boxed{u_i(10, a_j) > u_i(a_i', a_j)}$$

ITERATED STRICT DOMINANCE: For each $k \geq 0$

$$S_i^0 = A_i$$

$$S_i^1 = \{a_i \in S_i^0 : a_i \text{ not strictly dominated in } S_i^0 \times S_j^0\}$$

$$S_i^2 = \{a_i \in S_i^1 : a_i \text{ not strictly dominated in } S_i^1 \times S_j^1\}$$

$$\vdots$$

$$S_i^k = \{a_i \in S_i^{k-1} : a_i \text{ not strictly dominated in } S_i^{k-1} \times S_j^{k-1}\}$$

$$\vdots$$

$$\boxed{S_i = S_i^0 \cap S_i^1 \cap S_i^2 \cap \dots}$$

$$\boxed{ISD = S_a \times S_b}$$

	L	M	R
T	3, 3	3, 0	3, 2
M	2, 3	4, 0	2, 4
B	2, 4	2, 1	2, 5

$S_a^0 = \{T, M, B\}$, $S_b^0 = \{L, M, R\}$

$S_a^1 = \{T, M\}$, $S_b^1 = \{L, R\}$

$S_a^2 = \{T\}$, $S_b^2 = \{L, R\}$

$S_a^3 = \{T\}$, $S_b^3 = \{L\}$

$S_a^4 = \{T\}$, $S_b^4 = \{L\}$

$\vdots \qquad \vdots$

$$\boxed{\begin{aligned} ISD &= \{T\} \times \{L\} \\ &= \{(T, L)\} \end{aligned}}$$

WEAK DOMINANCE: a_i weakly dominates a_i' in the game $A_i \times A_j$, if for all $a_j \in A_j$:

$$\boxed{u_i(a_i, a_j) \geq u_i(a_i', a_j)}$$

with at least one inequality being strict

An action a_i' is weakly dominated if there exists $a_i \in A_i$ that weakly dominates

An action a_i is weakly dominant, if it weakly dominates every other $a_i' \in A_i$.

Question D.6: A weakly dominated action could be rational.

	L	R
T	1	1
B	1	0

B is weakly dominated by T.

Nevertheless, B could be rational, if we took beliefs $\mu_a(L) = 1$.

$$\mathbb{E}_{\mu_a}(u_a(T)) = \mathbb{E}_{\mu_a}(u_a(B)) \Rightarrow \text{both } T \text{ and } B \text{ are rational.}$$

(Question C.21) if $\mu_a(L) > 0$ and $\mu_a(R) > 0$ then only B is a rational action.

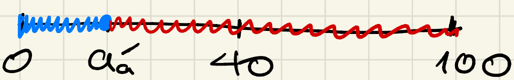
Question D.7 : Ann is selfish :

$$u_a(a_a, a_b) = \begin{cases} 40 - a_b, & a_a \geq a_b \\ 0, & a_a < a_b \end{cases}$$

$A_i = \{0, 1, \dots, 100\}$

$a_a = 40$ is a weakly dominant strategy.

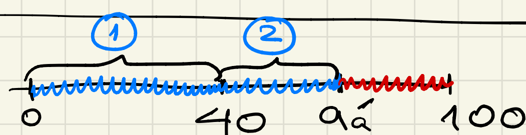
Case 1 : $a_a' < 40$



- If $a_b \leq a_a'$, Ann will win and she will get
  $u_a(a_a', a_b) = 40 - a_b$
  whereas if she bid $a_a = 40$
  $u_a(a_a, a_b) = 40 - a_b$.

- If $a_b > a_a'$, Ann will lose and she would get
  $u_a(a_a', a_b) = 0$
  whereas if she bid to $a_a = 40$
  $u_a(a_a, a_b) = 40 - a_b \geq 0, \quad a_b \leq 40$
  $u_a(a_a, a_b) = 0 \qquad\qquad\quad , a_b > 40$

Case 2 : $a_a' > 40$



- If $a_b \leq a_a'$, then Ann wins and she gets
  $u_a(a_a', a_b) = 40 - a_b$
  1 $\geq 0 \Rightarrow u_a(a_a, a_b) = 40 - a_b$
  2 $< 0 \Rightarrow u_a(a_a, a_b) = 0$

- If $a_b > a_a'$, then $u_a(a_a', a_b) = u(a_a, a_b) = 0$

Question D.8 : $\boxed{a_i \text{ weakly dominant} \Rightarrow a_i \text{ rational}}$

(Question D.4)

$\boxed{\text{For all } a_j \in A_j}$ $\boxed{\text{for all } a_i'}$

$u_i(a_i, a_j) \geq u_i(a_i', a_j) \Rightarrow$

$\Rightarrow \mu_i(a_j) u_i(a_i, a_j) \geq \mu_i(a_j) u_i(a_i', a_j)$

$\Rightarrow \sum_{a_j \in A_j} \mu_i(a_j) u_i(a_i, a_j) \geq \sum_{a_j \in A_j} \mu_i(a_j) u_i(a_i', a_j)$

$\Rightarrow \mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i'))$

$\Rightarrow a_i$ is rational.

ITERATED WEAK DOMINANCE : For each $k \geq 0$

$W_i^0 = A_i$

$W_i^1 = \{a_i \in W_i^0 : a_i \text{ not weakly dominated in } W_i^0 \times W_j^0\}$

$W_i^2 = \{a_i \in W_i^1 : a_i \text{ not weakly dominated in } W_i^1 \times W_j^1\}$

$\vdots$

$W_i^k = \{a_i \in W_i^{k-1} : a_i \text{ not weakly dominated in } W_i^{k-1} \times W_j^{k-1}\}$

$\vdots$

$\boxed{W_i = W_i^0 \cap W_i^1 \cap W_i^2 \cap \dots}$

$\boxed{IWD = W_a \times W_b}$

Question D.9:

(1)

	L	R
T	3, 1	3, 0
M	1, 1	2, 0
B	2, 1	1, 0

Dominant actions

(2)

	L	R
T	3, 1	2, 0
M	3, 2	3, 0
B	2, 0	1, 5

No dominant actions

(1) Ann chooses T, Bob chooses L.
Hence we immediately conclude that this is what they will play without considering how they have reasoned about their opponents

(2) Bob employs strategic reasoning when he discards Ann's action B on the basis of this action being "not rational".

Question D.10: $n = 100$, each asked to choose a number from $\{1, 2, \dots, 100\}$.
Compute the average and whoever is closer to 2/3 of this avg wins €100.

Take 67, and observe that any $a_i > 67$ is weakly dominated by 67. $\boxed{?}$

Thus everyone will figure this out.

So any action above 45 will be weakly dominated by 45.

Thus everyone will figure this out

⋮

Eventually the only action I am left with is 1.

BÖRGERS DOMINANCE:

Strict dominance: (too permissive) (Question D.3)

strictly dominated $\Rightarrow$ not rational

$\cancel{\Leftarrow}$

Weak dominance: (too restrictive) (Question D.6).

weakly dominated $\cancel{\Rightarrow}$ not rational

---

a_i is Börgers dominated in $A_i \times A_j$, if:

for every $A_j' \subseteq \boxed{A_j}$, I can find some a_i' that weakly dominates a_i in $A_i \times A_j'$

	L	R
T	10	0
M	0	10
B	0	0

No matter which actions of the opponent I look, an action is always weakly dominated:

$\{L\}$: B weakly dominated

$\{R\}$: B weakly dominated

$\{L, R\}$: B weakly dominated

$\boxed{\text{B is Börgers dominated}}$

	L	R
T	10	0
M	0	10
B	1	0

$\{L, R\}$: B weakly dominated

$\{L\}$: B weakly dominated

$\{R\}$: B weakly dominated

$\boxed{\text{B Börgers dominated}}$

	L	R
T	10	0
M	0	10
B	1	1

$\{L, R\}$: B <u>not</u> weakly dominated.

$\Longrightarrow$ $\boxed{\text{B not Börgers dominated}}$

<u>Theorem</u> :

a_i Börgers dominated $\Longleftrightarrow$ There is no Bernoulli utility function and no belief that makes a_i rational

ITERATED BÖRGERS DOMINANCE: $k \geq 0$

$B_i^0 = A_i$

$B_i^1 = \{a_i \in B_i^0 : a_i$ not Börgers dominated in $B_i^0 \times B_j^0\}$

$B_i^2 = \{a_i \in B_i^1 : a_i$ not Börgers dominated in $B_i^1 \times B_j^1\}$

$\vdots$

$B_i^k = \{a_i \in B_i^{k-1} : a_i$ not Börgers dominated in $B_i^{k-1} \times B_j^{k-1}\}$

$\vdots$

$$\boxed{B_i = B_i^0 \cap B_i^1 \cap B_i^2 \cap \cdots}$$

$$\boxed{IBS = B_a \times B_b}$$

Question D.11: $IWD \subseteq IBD \subseteq ISD$

Take any game and an action $a_i \in A_i$.

Suppose that a_i is Börgers dominated

$\Rightarrow a_i$ is weakly dominated for every $A_j' \subseteq A_j$

$\Rightarrow a_i$ is weakly dominated for A_j

$\boxed{a_i \text{ Börgers dominated} \Rightarrow a_i \text{ weakly dominated}}$

$\Updownarrow$

$\boxed{a_i \text{ not Börgers domin.} \Leftarrow a_i \text{ not weakly dominated}}$

a_i strictly dominated $\Rightarrow$

$\Rightarrow$ a_i is strictly dominated given every $A_j' \subseteq A_j$

$\Rightarrow$ a_i is weakly dominated given every $A_j' \subseteq A_j$

$\Rightarrow$ a_i Börgers dominated

$\boxed{a_i \text{ not Börgers dominated} \Rightarrow a_i \text{ not strictly dominated}}$

Elimination based on strategic reasoning

Equilibrium based on stability.

BEST RESPONSE : a_i is a best response to $\boxed{a_j}$

if for all $a_i' \in A_i$:

$$\boxed{u_i(a_i, \boxed{a_j}) \geq u_i(a_i', \boxed{a_j})}$$

$BR_i(a_j) \subseteq A_i$ all best responses to a_j.

Question D.12 :

(a) $\boxed{a_i \in BR_i(a_j) \text{ for some } a_j \Longrightarrow a_i \text{ not Börgers dominated.}}$

$u_i(a_i, a_j) \geq u_i(a_i', a_j) \Rightarrow$

$\Rightarrow \boxed{A_i \times \{a_j\}}$, action a_i is not weakly dominated.

still T weakly dominated by M

$\Rightarrow a_i$ not Börgers dominated.

	L	R
T	3	0
M	3	3
B	2	3

T is a best response to L

T not weakly dominated in the blue game.

So it is not Börgers dom.

(b) a_i strictly dominant $\Rightarrow$

$\Rightarrow u_i(a_i, a_j) > u_i(a_i', a_j)$ for all other a_i'

$\Rightarrow a_i$ best response to a_j

TRUE FOR ALL a_j.

(c) a_i weakly dominant $\Rightarrow$

$\Rightarrow u_i(a_i, a_j) \geq u_i(a_i', a_j)$ for all other a_i'

$\Rightarrow a_i$ best response to a_j

TRUE FOR all a_j.

	L	R
T	3	3
B	3	0

weakly dominant

B is also a BR to L.

NASH EQUILIBRIUM :

An action profile (a_a, a_b) such that

- $a_a \in BR_a(a_b)$ and
- $a_b \in BR_b(a_a)$

$$\left(a_i \in BR_i(a_j) \text{ for both } i \in I \right)$$

No player has an incentive to unilaterally deviate.

$NE \subseteq A_a \times A_b$: all Nash Equilibria

Question D.13 : $A_i = \{1,2,3,4\}$

	1	2	3	4
1	0, -6	0, 0	0, 0	0, 0
2	0, -12	4, 0	8, 0	8, 0
3	0, -12	0, 0	4, 2	8, 0
4	0, -12	0, 0	0, 4	0, 0

$$Q(p) = 16 - 4P$$

$$P = \min\{P_a, P_b\}$$

$$c_a = 1, \quad c_b = 2$$

$P_a = 1,\ P_b = 2$: $P = \min\{1,2\} = 1 \Rightarrow Q(1) = 12$

$$\Pi_a = 1 \cdot 12 - 1 \cdot 12 = 0$$

$$\Pi_b = 0$$

NE are the strategy profiles where they both best respond. $\boxed{NE = \{(2,2), (2,3), (2,4)\}}$

Question D.14:

	D	C
D	2,2	7,0
C	0,7	5,5

Prisoner's dilemma

NE = {(D,D)}

	O	B
O	3,1	0,0
B	0,0	1,3

Battle of the Sexes

NE = {(O,O), (B,B)}

	H	T
H	1,−1	−1,1
T	−1,1	1,−1

Matching Pennies

NE = $\emptyset$

Question D.15:

Question D.12(a) : $a_i \in BR(a_j) \Rightarrow$ not Börgers dominated.

$$(a_a, a_b) \in NE \iff \left.\begin{array}{l} a_a \in BR_a(a_b) \\ a_b \in BR_b(a_a) \end{array}\right\} \Rightarrow \text{both } a_a, a_b \text{ survive Börgers dominance.}$$

$\Rightarrow$ survive strict dominance

	L	R
T	10, 1	0, 0
M	0, 0	10, 1
B	5, 1	5, 0

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Question D.16 : Not all NE survive IWD

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

$$NE = \{(T,L), \boxed{(B,R)}\}$$

$$IWD = \{(T,L)\}$$

CARDINAL GAMES:

Everybody knows the Bernoulli utility function $\boxed{u_i : A_i \times A_j \longrightarrow \mathbb{R}}$

Player i's beliefs are still known only by player i.

MIXED ACTIONS: σ_i probability distribution over i's own actions ($\sigma_i(a_i)$ probability of action i occurring).

Each action a_i can also be seen as a mixed action $\sigma_i(a_i) = 1$.

EXPECTED UTILITY: of a mixed action σ_i given an action $a_j \in A_j$:

$$\boxed{u_i(\sigma_i, a_j) = \sum_{a_i \in A_i} \sigma_i(a_i) \cdot u_i(a_i, a_j)}$$

Subjective expected utility of σ_i given μ_i

$$\mathbb{E}_{\mu_i}(u_i(\sigma_i)) = \sum_{a_j \in A_j} \boxed{\mu_i(a_j)}\, \boxed{u_i(\sigma_i, a_j)}$$

$$= \sum_{a_j \in A_j} \underbrace{\mu_i(a_j)}_{\text{probabilities over opponent's actions}} \underbrace{\sum_{a_i \in A_i} \sigma_i(a_i)\, u_i(a_i, a_j)}_{\text{objective expected utility}}$$

Question D.17 : $\boxed{x \in \{4, 5, 6\}}$

$$
\begin{array}{c|c|c|c}
 & L & R & \\
\hline
T & 10 & 0 & (1/2) \\
M & 0 & 10 & (1/4) \\
B & x & x & (1/4) \\
\hline
\sigma_a & y_L & y_R & \\
 & \mu_a(L) & \mu_a(R) &
\end{array}
$$

$$\sigma_a = \left(\frac{1}{2} \times T, \frac{1}{4} \times M, \frac{1}{4} \times B\right)$$

$$u_a(\sigma_a, R) = \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 10 + \frac{1}{4} x = \boxed{\frac{1}{4}(10 + x)}$$

$$u_a(\sigma_a, L) = \frac{1}{2} \cdot 10 + \frac{1}{4} \cdot 0 + \frac{1}{4} x = \boxed{\frac{1}{4}(20 + x)}$$

$$\mu_a(L) = \frac{1}{4}, \quad \mu_a(R) = \frac{3}{4}$$

$$E_{\mu_a}(u_a(T)) = \frac{1}{4} \cdot 10 + \frac{3}{4} \cdot 0 = \frac{10}{4}$$

$$E_{\mu_a}(u_a(M)) = \frac{1}{4} \cdot 0 + \frac{3}{4} \cdot 10 = \frac{30}{4}$$

$$E_{\mu_a}(u_a(B)) = \frac{1}{4} \cdot x + \frac{3}{4} \cdot 10 = \frac{1}{4}(30 + x)$$

$$E_{\mu_a}(u_a(\sigma_a)) = \frac{1}{4} \cdot \boxed{\frac{1}{4}(20 + x)} + \frac{3}{4} \cdot \boxed{\frac{1}{4}(10 + x)}$$

$$= \frac{1}{16}(20 + x + 30 + 3x)$$

$$= \frac{1}{16}(50 + 4x)$$

RATIONAL CHOICE: A mixed strategy is rational in the game if it is the most preferred one

$\Longleftrightarrow$ a mixed strategy is rational if it yields the highest SEU (given u_i and μ_i).

(from the point of view of player i).

σ_i is rational: $\boxed{E_{\mu_i}(u(\sigma_i)) \geq E_{\mu_i}(u_i(\sigma_i')) \text{ for all } \sigma_i'}$

Support of mixed strategy

$\mathrm{supp}(\sigma_i)$: actions that receive positive probability

$$\mathrm{supp}(\sigma_i) = \{a_i \in A_i : \sigma_i(a_i) > 0\}.$$

Theorem:

$$\boxed{\sigma_i \text{ rational} \Longleftrightarrow E_{\mu_i}(u_i(a_i)) \geq E_{\mu_i}(u_i(a_i')) \text{ for all } a_i' \in A_i \text{ and for all } a_i \in \mathrm{supp}(\sigma_i)}$$

Example:

support of σ_a

$$\begin{array}{c|c|c|c}
 & L & R & \\ \hline
T & 10 & 0 & (1/2) \\ \hline
M & 0 & 10 & (1/2) \\ \hline
B & 4 & 4 & (0) \\ \hline
 & (1/2) & (1/2) &
\end{array}$$

$$\sigma_a = \left(\frac{1}{2} \times T, \frac{1}{2} \times M\right)$$

$$\mu_a(L) = \mu_a(R) = 1/2$$

$$\mathbb{E}_{\mu_a}(u_a(\sigma_a)) = \frac{1}{2} \cdot \frac{1}{2} 10 + \frac{1}{2} \cdot \frac{1}{2} 10 = 5$$

Take $\sigma_a' = (\alpha_1 \times T, \alpha_2 \times M, \alpha_3 \times B)$

$$\begin{aligned}
\mathbb{E}_{\mu_a}(u_a(\sigma_a')) &= \frac{1}{2}(10\alpha_1 + 4\alpha_3) + \frac{1}{2}(10\alpha_2 + 4\alpha_3) \\
&= 5(\alpha_1 + \alpha_2) + 2\alpha_3 \\
&= 5(\alpha_1 + \alpha_2) + 2(1 - (\alpha_1 + \alpha_2)) \\
&= 2 + 3(\alpha_1 + \alpha_2)
\end{aligned}$$

$$\mathbb{E}_{\mu_a}(u_a(\sigma_a)) = 5 \geq 2 + 3(\alpha_1 + \alpha_2) = \mathbb{E}_{\mu_a}(u_a(\sigma_a'))$$

σ_a is rational (given μ_a)

The actions in the support of σ_a to do better than the actions outside the support

$$\left.\begin{aligned}
\mathbb{E}_{\mu_a}(u_a(T)) &= \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 = 5 \\
\mathbb{E}_{\mu_a}(u_a(M)) &= \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 10 = 5
\end{aligned}\right\}$$

all actions within the support do equally good as one another.

$$\mathbb{E}_{\mu_a}(u_a(B)) = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 4 = 4$$

STRICT DOMINANCE: σ_i strictly dominates a_i whenever

$$\boxed{u_i(\sigma_i, a_j) > u_i(a_i, a_j)}$$

for all $a_j \in A_j$.

If there is some σ_i strictly dominating a_i, we say that a_i is strictly dominated.

Question D.18: $x \in \{4, 5, 6\}$

$$\begin{array}{c|c|c|c}
 & L & R & \\
\hline
T & 10 & 0 & \alpha \\
\hline
M & 0 & 10 & 1-\alpha \\
\hline
B & x & x & \\
\end{array}$$

- T not strictly dominated (for any x): Take any $\sigma_a = (\alpha \times M, (1-\alpha) \times B)$

$$u_a(\sigma_a, L) = (1-\alpha) \cdot x < 10 = u_a(T, L)$$

- M not strictly dominated (for any x)

$$\sigma_a = (\alpha \times T, (1-\alpha) \times B)$$
$$u_a(\sigma_a, R) = (1-\alpha) \cdot x < 10 = u_a(M, R)$$

- Take $\sigma_a = (\alpha \times T, (1-\alpha) \times M)$

$$u_a(\sigma_a, L) = \boxed{10\alpha > x} = u_a(B, L)$$
$$u_a(\sigma_a, R) = \boxed{10 - 10\alpha > x} = u_a(B, R)$$

QUESTION D2

B rational when $x \geq 5$

B not rational when $x = 4$

(i) $\boxed{x = 4}$: $\left.\begin{array}{l} 10\alpha > 4 \iff \alpha > 4/10 \\ 10 - 10\alpha > 4 \iff \alpha < 6/10 \end{array}\right\}$ $\boxed{\frac{4}{10} < \alpha < \frac{6}{10}}$

(ii) $\boxed{x \geq 5}$: $\left.\begin{array}{l} 10\alpha > 5 \iff \alpha > 1/2 \\ 10 - 10\alpha > 5 \iff \alpha < 1/2 \end{array}\right\}$ cannot occur simultaneously

Theorem: Fix Bernoulli utility u_i:

a_i strictly dominated (by some σ_i) $\Longleftrightarrow$ a_i not rational (for any belief)

Question D.19:

a_i not strictly dominated (by any σ_i) $\Longleftrightarrow$ a_i is rational (for some belief)

Question D.20: see previous page

Question D.21:

	L	R
T	3	3
B	2	2

If these utilities simply represent preferences over outcomes, no matter which Bernoulli utility function I take, it will always be the case that T strictly dominates B.

	L	R
T	10	0
M	0	10
B	4	4

If these are Bernoulli utilities B is strictly dominated, but not in the ordinal game. If I replace 4 with 6, I will get same ordinal preferences, and B will not be dom. $\boxed{6}$

Question D.22:

	L	R
T	10	0
M	0	10
B	x	x

No matter which $x \in \{4,5,6\}$ I select, Börgers dominance gives me always the same answer, viz., B not Börgers dominated.

$\iff$ B is rational for some u_i (take $x = 6$).

Suppose I know Bernoulli utilities of Ann (in particular I know $x=4$).
In this case B strictly dominated

$\iff$ B not rational for given u_i

ITERATED STRICT DOMINANCE: For $k \geq 0$

$$S_i^0 = A_i$$

$$S_i^1 = \{a_i \in S_i^0 : a_i \text{ not strictly dominated in } S_i^0 \times S_j^0\}$$

$$S_i^2 = \{a_i \in S_i^1 : a_i \text{ not strictly dominated in } S_i^1 \times S_j^1\}$$

$$\vdots$$

$$S_i^k = \{a_i \in S_i^{k-1} : a_i \text{ not strictly dominated in } S_i^{k-1} \times S_j^{k-1}\}$$

$$\vdots$$

$$\boxed{S_i = S_i^0 \cap S_i^1 \cap S_i^2 \cap \cdots}$$

$$\boxed{ISD = S_a \times S_b}$$

Question D.23 : $A_i = \{2, \dots, 6\}$ $u(x) = x$

	2	3	4	5	6
2	2, 2	4, 0	4, 0	4, 0	4, 0
3	0, 4	3, 3	5, 1	5, 1	5, 1
4	0, 4	1, 5	4, 4	6, 2	6, 2
5	0, 4	1, 5	2, 6	5, 5	7, 3
6	0, 4	1, 5	2, 6	3, 7	6, 6

No action is strictly dominated by another action.

$ISD = \{(2,2)\}$

BEST RESPONSE: A mixed action σ_i is a best response to σ_j, if for all σ_i':

$$\boxed{\mathbb{E}_{\sigma_j}(u(\sigma_i)) \geq \mathbb{E}_{\sigma_j}(u_i(\sigma_i'))}$$

We write $\sigma_i \in BR_i(\sigma_j)$

i responds optimally j's mixed action

$\iff$ i responds optimally to his beliefs

AND these belief turn out to be correct

Question D.24:

Thm D.2: Given belief μ_i

σ_i is rational $\iff$ a_i rational for all $a_i \in \mathrm{supp}(\sigma_i)$

$$\sigma_i \in BR_i(\sigma_j) \iff \mathbb{E}_{\mu_i}(u_i(\sigma_i)) \geq \mathbb{E}_{\mu_i}(u_i(\sigma_i'))$$

for all σ_i', **AND**

$\mu_i = \sigma_j$

$$\overset{\text{Thm D.2}}{\iff} \mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i'))$$

for all $a_i' \in A_i$ and all

$a_i \in \mathrm{supp}(\sigma_i)$, **AND** $\mu_i = \sigma_j$

$$\iff \mathbb{E}_{\sigma_j}(u_i(a_i)) \geq \mathbb{E}_{\sigma_j}(u_i(a_i'))$$

for all $a_i' \in A_i$ and $a_i \in \mathrm{supp}(\sigma_i)$

NASH EQUILIBRIUM: (σ_a, σ_b) is a NE if $\sigma_a \in BR_a(\sigma_b)$ and $\sigma_b \in BR_b(\sigma_a)$

I write $(\sigma_a, \sigma_b) \in NE$.

Theorem. Every game has a NE

Question D.25: $(\sigma_a, \sigma_b) \in NE \Rightarrow a_i$ survives ISD for all $a_i \in supp(\sigma_i)$ and all $i \in I$.

$\sigma_i \in BR_i(\sigma_j) \Rightarrow$

$$\mathbb{E}_{\boxed{\sigma_j}}(u_i(a_i)) \geq \mathbb{E}_{\boxed{\sigma_j}}(u_i(a_i'))$$

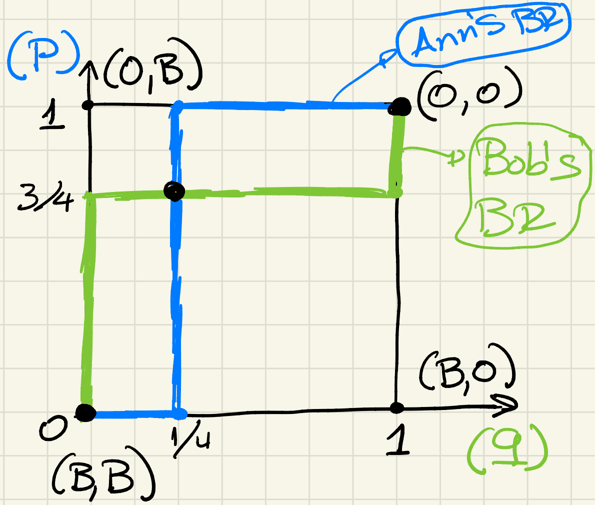
for all $a_i' \in A$. $\Rightarrow$

$\Rightarrow a_i$ is rational

$\Rightarrow a_i$ not strictly dominated

Question D.26 :

	O (q)	B ($1-q$)	
O	3, 1	0, 0	p
B	0, 0	1, 3	$1-p$



$$\sigma_a = (p \times O, (1-p) \times B)$$

$$\sigma_b = (q \times O, (1-q) \times B)$$

Fix some $q \in [0,1]$:

$$\left.\begin{aligned} \mathbb{E}_q(u_a(O)) &= 3q \\ \mathbb{E}_q(u_a(B)) &= 1-q \end{aligned}\right\} \Rightarrow \quad p=1 \iff \mathbb{E}_q(u_a(O)) \geq \mathbb{E}_q(u_a(B)) \iff 3q \geq 1-q \iff q \geq \tfrac{1}{4}$$

$$p=0 \iff q \leq 1/4$$

$$0 \leq p \leq 1 \iff q = 1/4$$

Fix some $p \in [0,1]$:

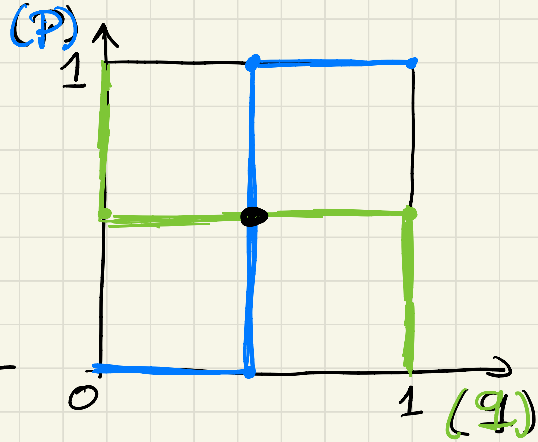
$$\left.\begin{aligned} \mathbb{E}_p(u_b(O)) &= p \\ \mathbb{E}_p(u_b(B)) &= 3-3p \end{aligned}\right\} \Rightarrow \quad q=1 \iff \mathbb{E}_p(u_b(O)) \geq \mathbb{E}_p(u_b(B)) \iff p \geq 3-3p \iff p \geq 3/4$$

$$q=0 \iff p \leq 3/4$$

$$0 \leq q \leq 1 \iff p = 3/4$$

$$NE = \{(0,0), (\tfrac{1}{4}, \tfrac{3}{4}), (1,1)\}$$

$$
\begin{array}{c|cc|c}
 & H & T & \\
\hline
H & 1,-1 & -1,1 & p \\
T & -1,1 & 1,-1 & 1-p \\
\hline
 & q & 1-q &
\end{array}
$$



Fix $q \in [0,1]$:

$$\left.\begin{aligned} \mathbb{E}_q(u_a(H)) &= q-(1-q) = 2q-1 \\ \mathbb{E}_q(u_a(T)) &= -q+(1-q) = -2q+1 \end{aligned}\right\} \Rightarrow p=1 \iff 2q-1 \geq -2q+1 \iff q \geq 1/2$$

$p=0 \iff q \leq 1/2$

$0 \leq p \leq 1 \iff q = 1/2$

Fix $p \in [0,1]$:

$$\left.\begin{aligned} \mathbb{E}_p(u_b(H)) &= -p+(1-p) = -2p+1 \\ \mathbb{E}_p(u_b(T)) &= p-(1-p) = 2p-1 \end{aligned}\right\} \Rightarrow q=1 \iff -2p+1 \geq 2p-1 \iff p \leq \frac{1}{2}$$

$q=0 \iff p \geq 1/2$

$0 \leq q \leq 1 \iff p = 1/2$

$$NE = \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\}$$

Question D.27:

$$\begin{array}{c|c|c|l} & H & T & \\ \hline H & 1,\ -1 & -1,\ 1 & (49) \\ \hline T & -1,\ 1 & 1,\ -1 & (51) \\ \hline & (47) & (53) & \end{array} \quad \sigma_a = (0.49 \times H, 0.51 \times T)$$

interpret it as
a mixed action
of Bob

$\sigma_b = (0.47 \times H, 0.53 \times T)$

Different observations are independent

Question D.28:

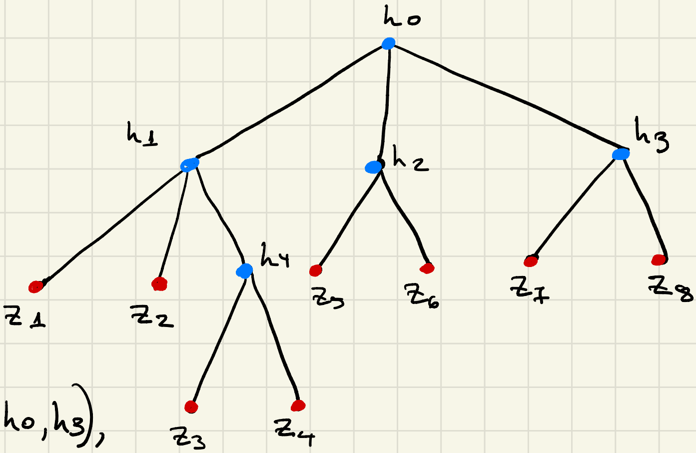
Cardinal games strictly increasing linear transformations of Bernoulli utilities do not affect NE and ISD.
(Question C.8)

Ordinal games strictly increasing transformations of utilities do not affect NE, ISD, IBD, IWD.
(Question A.3)

EXTENSIVE FORM GAMES:

GAME TREE: A (directed) graph

(Question E.1)



$(\bar{H}, \mathcal{E})$ where

$\bar{H}$: set of nodes

$\bar{H} = \{\underbrace{h_0, \dots, h_4}_{\text{non-terminal}}, \underbrace{z_1, \dots, z_8}_{\text{terminal}}\}$

$\mathcal{E}$: set of edges

$\mathcal{E} = \{(h_0,h_1), (h_0,h_2), (h_0,h_3),$
$(h_1,h_4), (h_1,z_1), (h_1,z_2),$
$(h_4,z_3), (h_4,z_4), (h_2,z_5),$
$(h_2,z_6), (h_3,z_7), (h_3,z_8)\}$

$$\boxed{\begin{gathered} H = \{h_0, h_1, h_2, h_3, h_4\} \\ Z = \{z_1, \dots, z_8\} \end{gathered}}$$

When $(h, h') \in \mathcal{E}$ I say that
h is a direct predecessor of h' and
h' is a direct successor of h.

Each $h' \in \bar{H}$ has at most one direct predecessor
(h_0 has no direct predecessors: the root)
(every other node has exactly one direct predecessor)

Nodes without direct successors are called terminal and are denoted by Z. Other nodes are non-terminal and are denoted by H.

Successors: $(h_1, \dots, h_N)$, then h_N is a successor h_1

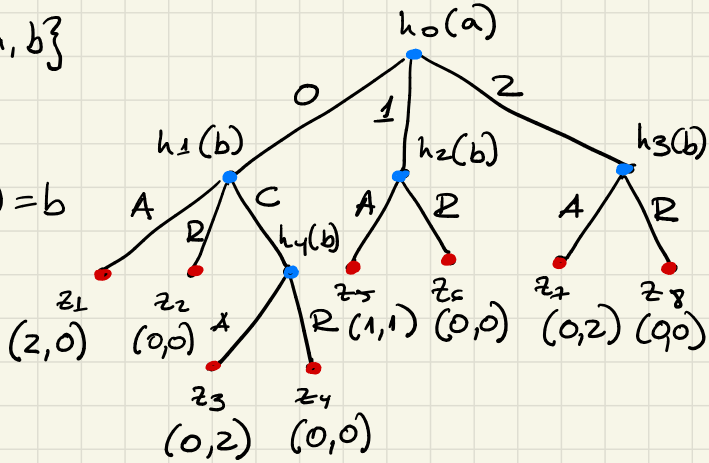
Predecessors: h_1 is a predecessor of h_N

Active players: $I = \{a, b\}$

$P : H \to I$

$P(h_0) = a$, $P(h_1) = b$, $P(h_2) = b$

$P(h_3) = b$, $P(h_4) = a$



Actions at each non-terminal node:

$A_i(h)$: actions of active player i at node h.

$A_a(h_0) = \{0, 1, 2\}$, $A_b(h_1) = \{A, R, C\}$, $A_b(h_2) = \{A, R\}$

$A_b(h_3) = \{A, R\}$, $A_a(h_4) = \{A, R\}$.

Outcomes: $o_i : Z \to \mathbb{R}$, $(\underbrace{o_a(z)}_{\text{Anni's payoff}}, \underbrace{o_b(z)}_{\text{Bob's payoff}})$

$(2,0) \succsim_a (1,1)$
$\Updownarrow$
$z_1 \succsim_a z_5$

Preferences: $\succsim_i$ over payoff vectors $\Rightarrow$ preferences over Z.

Ordinal utilities: u_i representing $\succsim_i$

$$\boxed{z \succsim_i z' \iff u_i(z) \geq u_i(z')}$$

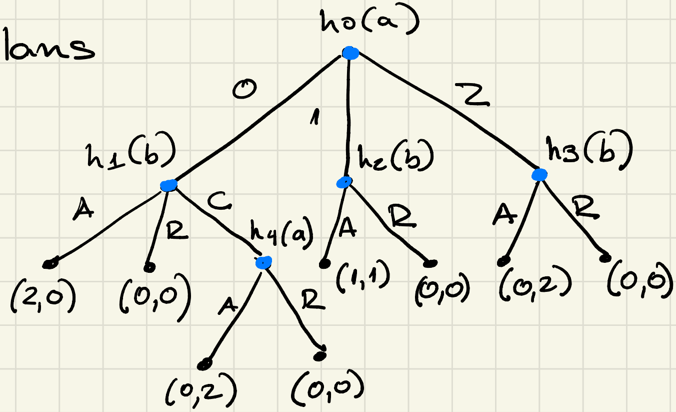
Game is without ties if $u_i(z) \neq u_i(z')$ for all z, z'.

Strategies: complete plans of action.

$$\boxed{S_i = \bigtimes_{h \in H_i} A_i(h)} \quad (*)$$

(H_i: nodes in H where i active)

$$s_i = (\underbrace{s_i(h)}_{\text{the action prescribed by strategy } s_i \text{ at } h.})_{h \in H_i}$$



Question E.2:

$$S_a = \{\overbrace{\underbrace{0}_{s_a(h_0)}\underbrace{A}_{s_a(h_4)}}^{s_a}, 0R, 1A, \boxed{1R}, 2A, 2R\} = \underbrace{\{0,1,2\}}_{A_a(h_0)} \times \underbrace{\{A,R\}}_{A_a(h_4)}$$

"At h_0 choose 1 and if you find yourself at h_4 choose R".

$$S_b = \{AAA, AAR, ARA, ARR, RAA, RAR, RRA, RRR, \overbrace{\underbrace{C}_{s_b(h_1)}\underbrace{A}_{s_b(h_2)}\underbrace{A}_{s_b(h_3)}}^{s_b}, CAR, CRA, CRR\} = \underbrace{\{A,R,C\}}_{A_b(h_1)} \times \underbrace{\{A,R\}}_{A_b(h_2)} \times \underbrace{\{A,R\}}_{A_b(h_3)}$$

nodes consistent with a strategy: $s_i \in S_i$, then $H(s_i)$ nodes that can be reached.

Example: $H(1R) = \{h_0, h_2\}$

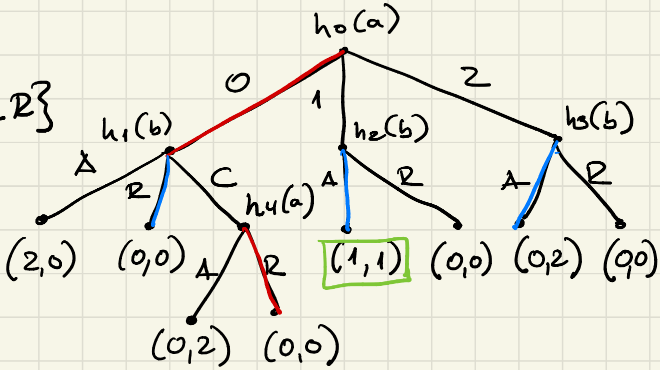
$$H_i(s_i) = H_i \cap H(s_i)$$

Example: $H_a(1R) = \{h_0\}$

Question E.3 :

$S_a = \{0A, 0R, 1A, 1R, 2A, 2R\}$

$S_b = \{AAA, AAR, ARA, ARR,$
$RAA, RAR, RRA, RRR,$
$CAA, CAR, CRA, CRR\}$



$H(0A) = \{h_0, h_1, h_4\}$ $\quad H_a(0A) = \{h_0, h_4\}$

$H(2R) = \{h_0, h_3\}$ $\quad H_a(2R) = \{h_0\}$

$H(ARA) = \{h_0, h_1, h_2, h_3\}$ $\quad H_b(ARA) = \{h_1, h_2, h_3\}$

$H(CAA) = \{h_0, h_1, h_2, h_3, h_4\}$ $\quad H_b(CAA) = \{h_1, h_2, h_3\}$

Question E4.

$(0R, RAA) \longmapsto z_2 \longmapsto (\boxed{0}, 0)$

$(0R, AAR) \longmapsto z_1 \longmapsto (\boxed{2}, 0)$

$(1A, AAA) \longmapsto \boxed{z_5}$

$(1R, CAR) \longmapsto \boxed{z_5}$

$\boxed{?}$

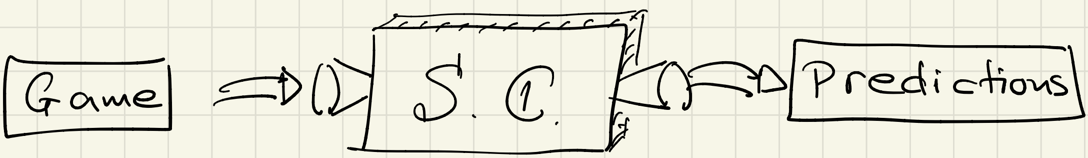
INFORMATION : observations about past moves

- Perfect information
- Imperfect information

SOLUTION CONCEPTS :

	ELIMINATION	EQUILIBRIUM
BEGINNING	—	Nash Equilibrium
THROUGHOUT THE ENTIRE GAME	Backward induction	Subgame Perfect Equilibrium

Question E.S :



Predictions in terms of strategy profiles (in strategic form).

Sometimes predictions are made in terms of outcomes / terminal nodes

Strategy · method : report their entire strategy.

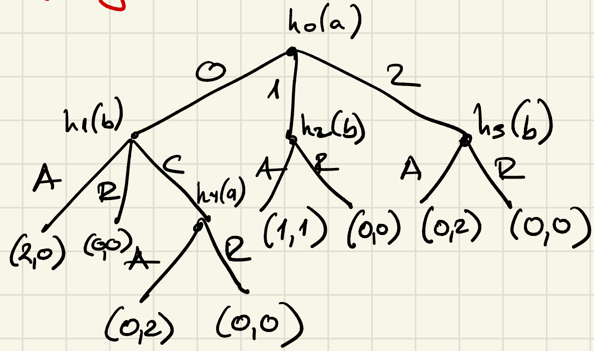
PERFECT INFORMATION GAMES:

Corresponding strategic form:

- Take extensive form game
- Find the strategies of each player
- Find the terminal node of each strategy profile
- Write it as a strategic form game:

Essentially we remove information about the dynamic aspect of the game.

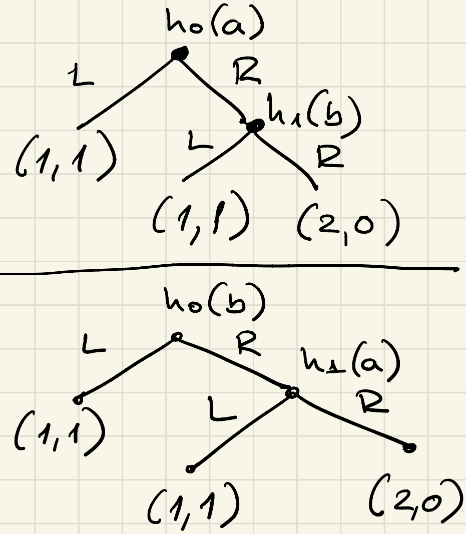
Question E.6:



	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
0A	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,2	0,2	0,2	0,2
0R	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
1A	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
1R	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
2A	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0
2R	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

Question E.7 :

	L	R
L	1,1	1,1
R	1,1	2,0



NASH EQUILIBRIUM : NE in the corresponding strategic form

Pure strategy NE as we only consider ordinal utilities.

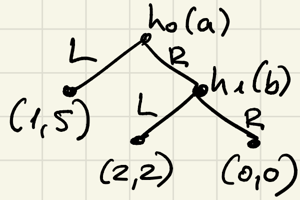
Question E.8 : See previous page.

Question E.9 : See above

Question E.10 :

NE = {LR, RL}

	L	R
L	1,5	1,5
R	2,2	0,0



Corresponding strategic form at h:

$S_i(h) \subseteq S_i$: strategies of player i which are consistent with h.

BACKWARD INDUCTION: For $k \geq 0$ and $h \in H_i$:

$$S_i^0(h) = S_i(h)$$

$$S_i^1(h) = \{ s_i \in S_i^0(h) : s_i \text{ not strictly dominated in } S_i^0(h') \times S_j^0(h') \text{ for any } h' \text{ that weakly succeeds } h \}$$

$$S_i^2(h) = \{ s_i \in S_i^1(h) : s_i \text{ not strictly dominated in } S_i^1(h') \times S_j^1(h') \text{ for any } h' \text{ that weakly succeeds } h \}$$

$\vdots$

$$S_i^k(h) = \{ s_i \in S_i^{k-1}(h) : s_i \text{ not strictly dominated in } S_i^{k-1}(h') \times S_j^{k-1}(h') \text{ for any } h' \text{ that weakly succeeds } h \}$$

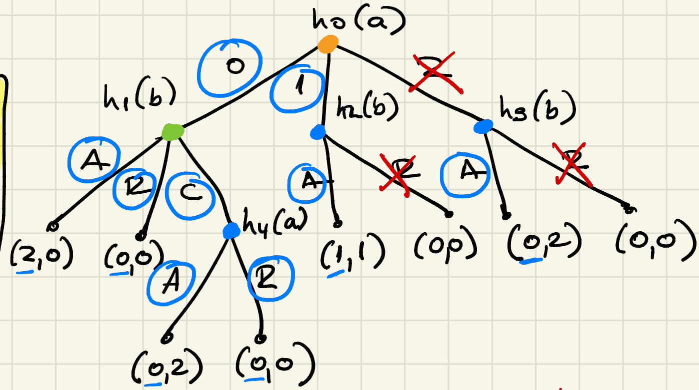
$\vdots$

$$\boxed{S_i^\infty(h_0) = S_i^0(h_0) \cap S_i^1(h_0) \cap S_i^2(h_0) \cap \cdots}$$

$$\boxed{BI = S_a^\infty \times S_b^\infty}$$

Question E.11 :

$$\boxed{\begin{array}{l} BI = \{0A, 0R, 1A, 1R\} \\ \times \{AAA, RAA, CAA\} \end{array}}$$



h0	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
0A	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,2	0,2	0,2	0,2
0R	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
1A	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
1R	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
~~2A~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~
~~2R~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~	~~0,2~~	~~0,0~~

h1	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
0A	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,2	0,2	0,2	0,2
0R	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0

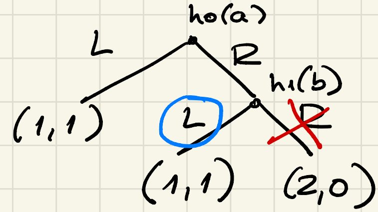
h2	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
1A	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
1R	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0

h3	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
2A	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0
2R	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

h4	CAA	CAR	CRA	CRR
0A	0,2	0,2	0,2	0,2
0R	0,0	0,0	0,0	0,0

h_0	L	R
L	1,1	1,1
R	1,1	2,0

h_1	L	R
R	1,1	2,0

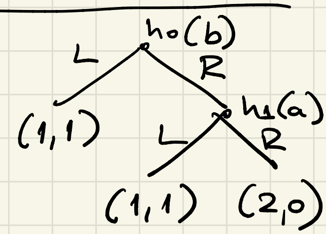


BI = {(L,L), (R,L)}

---

h_0	L	R
L	1,1	1,1
R	1,1	2,0

h_1	R
L	1,1
R	2,0

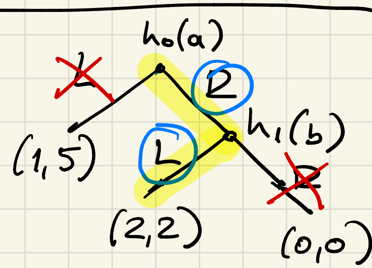


BI = {(R,L)}

---

h_0	L	R
L	1,5	1,5
R	2,2	0,0

h_1	L	R
R	2,2	0,0



BI = {(R,L)}

NE = {(L,R), (R,L)}

unintuintive

BI selects the more intuitive NE

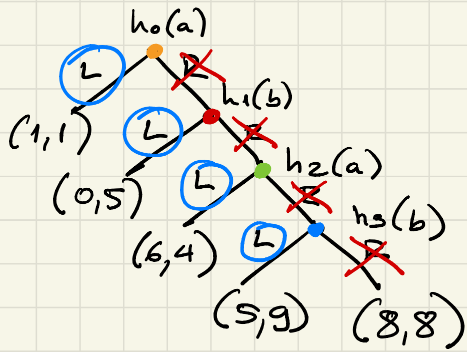
Question E.12 : see previous page

$\boxed{\text{Theorem: If the game has no ties, then there is a \underline{unique} strategy profile surviving BI.}}$

Question E.13: $BI \not\subseteq NE$ nor $NE \not\subseteq BI$
(see previous page).

Without ties $BI \subseteq NE$
(see previous page).

Question E.14:

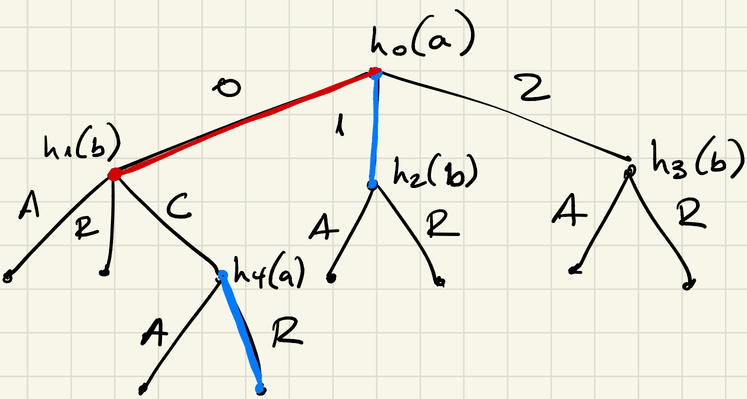


$BI = \{(LL, LL)\}$

CONTINUATION STRATEGIES:

Take $s_i \in S_i$. If $h \in H(s_i)$, then $s_i^h = s_i$.

If $h \in H(s_i)$, then $s_i^h(h')$ prescribes the move that an alternative strategy would have told me to do, and this alternative strategy leads me to h at preceding nodes and agrees with s_i from h onwards.



Question E.15: $s_a = \boxed{1R}$

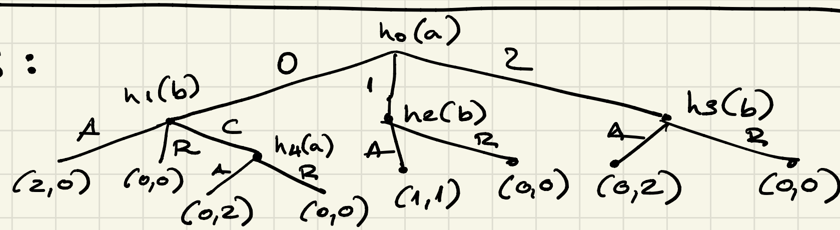
- At h_0, the continuation strategy of s_a is s_a.
- At $h_4 \notin H(1R) = \{h_0, h_2\}$.

$s_a^{h_4} = 0\underline{R}$

SUBGAME PERFECT EQUILIBRIUM: (S_a, S_b) such that (S_a^h, S_b^h) is a Nash Equilibrium at every $h \in H$.

Subgame Perfect Equilibrium (SPE) remains a NE throughout the game, even after possible mistakes. $\boxed{\text{NE robust to mistakes}}$

Question E.16:



ho	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
0A	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,2	0,2	0,2	0,2
0R	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
1A	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
1R	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
2A	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0
2R	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

h1	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
0A	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,2	0,2	0,2	0,2
0R	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0

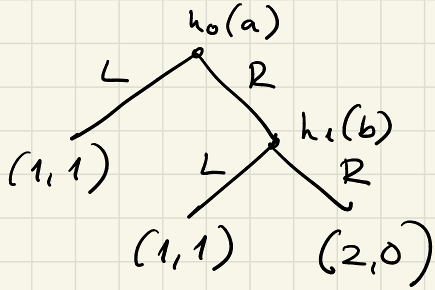
h2	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
1A	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
1R	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0

h3	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR	CAA	CAR	CRA	CRR
2A	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0
2R	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

h4	CAA	CAR	CRA	CRR
0A	0,2	0,2	0,2	0,2
0R	0,0	0,0	0,0	0,0

h_0	L	R
L	1,1	1,1
R	1,1	2,0

h_1	L	R
R	1,1	2,0



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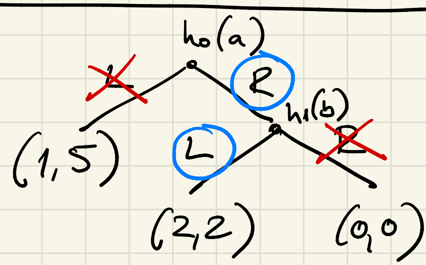
h_0	L	R
L	1,5	1,5
R	2,2	0,0

h_1	L	R
R	2,2	0,0



$\boxed{\text{Theorem : SPE survives BI}}$

Question E.17: Without ties, there is a unique BI strategy profile (Thm E.1). This is the unique SPE.

Question E.18: ULTIMATUM GAME:

Take any strategy of Bob. This will be a vector of A's and R's:

- NE: • Ann will offer the minimum that Bob accepts (if Bob accepts something).
  - Ann offers 0 and Bob rejects everything.
- SPE: • Bob accepts everything (including 0), Ann offers him 0.
  - Bob accepts everything (except 0), Ann offers him 1.

TRUST GAME:

- **NE**: In all NE both players simultaneausly best respond. Ann will offer the minimum amount to which Bob sends at least the same amount back. Bob never sends money back.

- **SPE**: Bob will never send anything back. Ann anticipates this and never sends him anything in the first place.

IMPERFECT INFORMATION: Players may not observe some of actions that have taken place.

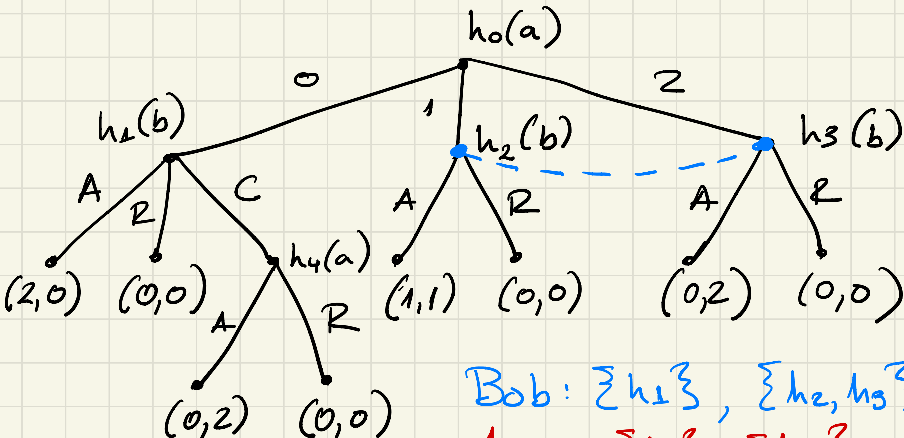
Information sets: $I_i \subseteq H_i$

all nodes in I_i are indistinguishable by i while being at them.

In fact the indinguishable also indirectly: $A_i(h) = A_i(h')$ for all $h, h' \in I_i$.

$\mathcal{I}_i$ the collection of all information sets.

Perfect information: $\{h\}$ is the information set at h (i knows at which node he is at)



Bob: $\{h_1\}$, $\{h_2, h_3\}$

Ann: $\{h_0\}$, $\{h_4\}$.

h_2 and h_3 are indistinguishable by Bob.

$\Longleftrightarrow$ Bob (at h_2 or h_3) does not know if Ann at h_0 chose 1 or 2.

Question E.19: Bob: $\{h_1\}, \{h_2\}, \{h_3\}$ / Ann $\{h_0\}, \{h_4\}$

Question E.20: $h, h' \in I_i \Rightarrow$ the same action is chosen by i.

$$\boxed{\text{Players condition their action on the information available to them}}$$

STRATEGIES: rewrite strategies s_i assigns an action to each information

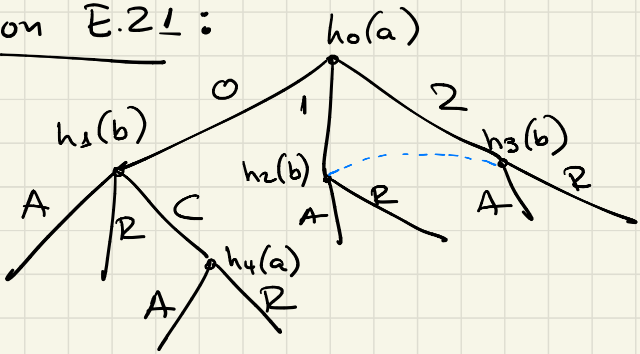
$s_i(I_i)$ instead of $s_i(h)$

$\Leftrightarrow s_i(h)$ $+$ $s_i(h) = s_i(h')$ for all $h, h' \in I_i$.

$A_i(I_i)$: action available at I_i.

$$\boxed{S'_i = \bigtimes_{I_i \in \mathcal{I}_i} A_i(I_i)}$$

Question E.21:



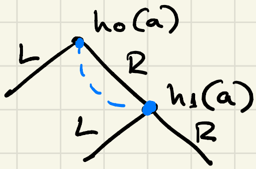
$$S'_a = \{0, 1, 2\} \times \{A, R\} = \{0A, 0R, 1A, 1R, 2A, 2R\}$$

$$S'_b = \underbrace{\{A, R, C\}}_{\text{at } \{h_1\}} \times \underbrace{\{A, R\}}_{\text{at } \{h_2, h_3\}} = \{A\boxed{A}, AR, RA, RR, CA, CR\}$$

what Bob plans to do at $\{h_2, h_3\}$

PERFECT RECALL: Players remember what they have earlier done themselves, and what they previously knew.

Question E.22: Absentmindedness:

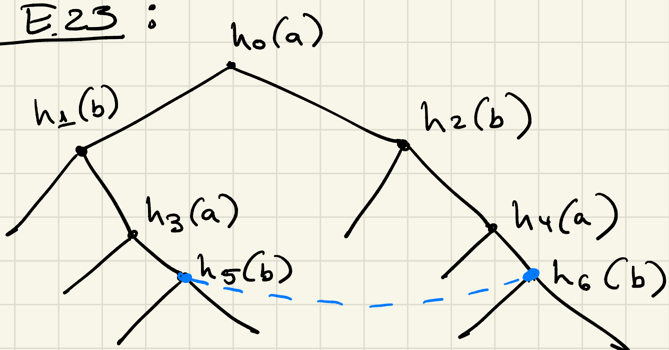


$I_a = \{h_0, h_1\} \Rightarrow$

if $s_i(I_a) = R$ then at h_1 she does not know if h_0 has already happened or not.

$\Rightarrow$ she cannot figure out that she has already played R.

Question E.23:



At $\{h_5, h_6\}$ Bob does not know what Ann played at h_0. But this is something he knew in the past (namely at $\{h_1\}$ and $\{h_2\}$)

So he violates recall of past knowledge.

SUBGAME: trimmed game tree

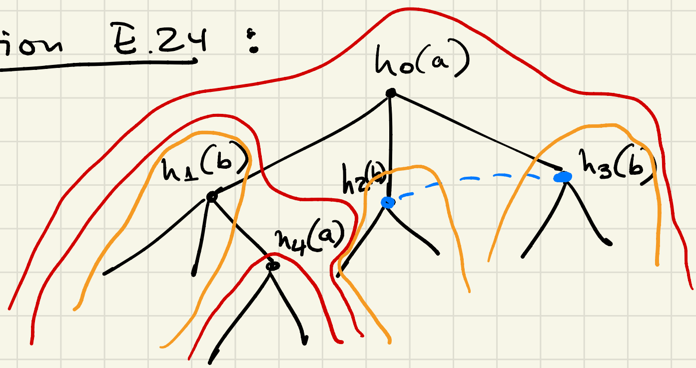
I can delete some nodes, but still:

(a) there is a root

(b) I don't break information sets

(c) all successors will also belong to the subgame

Question E.24:



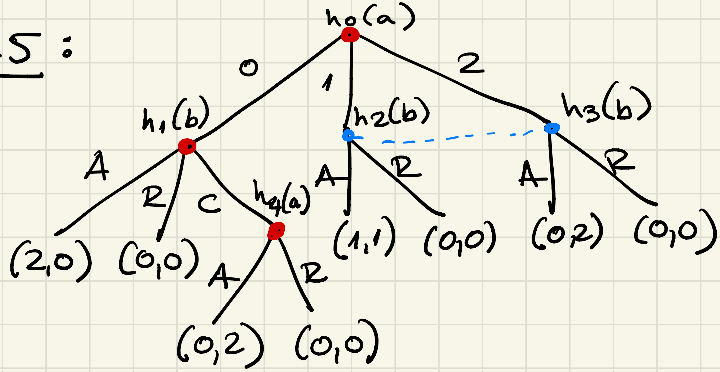
The red games are subgames
The yellow ones are not

Corresponding strategic form: I can write it for every subgame in exactly the same way.

NE: Nash equilibrium at the subgame that starts at h_0

SPE: Every continuation strategy to be a NE at every subgame.

Question E.25:



h0	AA	AR	RA	RR	CA	CR
0A	2,0	2,0	0,0	0,0	0,2	0,2
0R	2,0	2,0	0,0	0,0	0,0	0,0
1A	1,1	0,0	1,1	0,0	1,1	0,0
1R	1,1	0,0	1,1	0,0	1,1	0,0
2A	0,2	0,0	0,2	0,0	0,2	0,0
2R	0,2	0,0	0,2	0,0	0,2	0,0

h1	AA	AR	RA	RR	CA	CR
0A	2,0	2,0	0,0	0,0	0,2	0,2
0R	2,0	2,0	0,0	0,0	0,0	0,0

h4	CA	CR
0A	0,2	0,2
0R	0,0	0,0